國立臺灣大學數學系

九十七學年度博士班入學考試試題

科目：數值偏微分方程

2008.04.25

1. (20 points) Suppose that we want to use the linear multistep method (LMM) of the form

$$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = \Delta t(\beta_0 f(U^n) + \beta_1 f(U^{n+1}))$$

to find numerical approximation of the following initial value problem:

$$u'(t) = f(u), \qquad u(t_0) = u_0, \qquad t > t_0,$$

where $f \in \mathbb{R}$ is a prescribed Lipschitz continuous function in u over some domain, U^n is the approximate solution of u at time t_n, and Δt is the time step.

(a) For what values of β_0 and β_1 is local truncation error $\mathcal{O}((\Delta t)^2)$?

(b) Suppose you use the values of β_0 and β_1 just determined in this LMM. Is this a convergent method?

2. (20 points) Consider the following method for solving the heat equation $u_t = u_{xx}$ with suitable initial and boundary datas over some domain:

$$U_j^{n+2} = U_j^n + \frac{2\Delta t}{(\Delta x)^2}(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}).$$

Here U_j^n is the approximate solution of u at the spatial location x_j and time t_n, Δt and Δx are the temporal and spatial discretization sizes, respectively.

(a) Determine the order of accuracy of this method (in both space and time).

(b) Suppose we take $\Delta t = \alpha(\Delta x)^2$ for some fixed $\alpha > 0$ and refine the grid. For what values of α (if any) will this method be Lax-Richtmyer stable and hence convergent?

3. (35 points) Consider the method

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1}) \tag{1}$$

for the advection equation $u_t + au_x = 0$ on $0 \le x \le 1$ with initial condition $u(x, t_0) = u_0(x)$ and periodic boundary conditions. Here as before U_j^n represents the approximate solution of u at the spatial location x_j and time t_n, Δt and Δx are the temporal and spatial discretization sizes, respectively.

(a) Suppose that we view this method as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the advection equation. What is the matrix A?

(b) Suppose we want to fix the Courant number $\nu = a\Delta t/\Delta x$ as $\Delta t,\ \Delta x \to 0$. For what range of ν will the method be stable if $a > 0$? If $a < 0$?

(c) Suppose we use the same method (1) for the initial-boundary value problem with $u(0,t) = g_0(t)$ specified. Since the method has a one-sided stencil, no numerical boundary condition is needed at the right boundary. For what range of $\nu = a\Delta t/\Delta x$ will the CFL condition be satisfied in this case? What are the eigenvalues of the A matrix for this case and when will the method be stable?

4. (25 points) Devise a consistent and stable finite difference method to solve the biharmonic equation $\nabla^4 u = f$ over a rectangular domain D in two space dimensions with the Dirichlet boundary condition $u|_{\partial D} = g$. Here f and g are some prescribed functions.