國立臺灣大學數學系
九十六學年度博士班入學考試試題
科目：數值偏微分方程

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0) Please answer the following as clearly as possible, including any expansion, matrix, scheme and proof.

1) Given a function $f : [a, b] \to R$, a Newton quadrature starts with partitioning $[a, b]$ into N equal pieces with length $h = (b-a)/N$ and approximating f piecewise by polynomials for integration's sake.
(a) State the order of consistency of such a procedure in terms of h.
(b) If f is $(b-a)$-periodic and smooth, show the best possible consistency of the Riemann sum $\int_a^b f\,dx \approx h\,\Sigma_{i=1}^N f(a+i\,h)$.

2) Design a finite-difference or finite-element scheme to discretize the boundary value problem

$$u_{xx} = f,\ 0 \le x \le 1 \ \text{ and } \ u(0) - u_x(0) = 0 = u(1) + u_x(1).$$

Show (a)the consistency and (b)the solvability of your scheme.

3) Do the same things as asked in 2) but the BVP is now given by

$$u_{xxxx} = f,\ 0 \le x \le 1 \ \text{ and } \ u_x(0) - u_{xx}(0) = 0 = u_x(1) + u_{xx}(1).$$

4) Consider the following initial boundary value problem

$$u_t + \sin(x)u_x = 0, \quad 0 \le x \le 2\pi,$$

$$u(x,0) = \sin(x), \quad u(0,t) = u(2\pi,t), \quad t \ge 0.$$

(a) Determine the characteristics of this system.
(b) Show for each x fixed the asymptotic behavior of $u(x,t)$ as $t \to \infty$.
(c) Design a numerical scheme of the form

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \sin(j\Delta x)\frac{\theta_j(U_j^n - U_{j-1}^n) + (1-\theta_j)(U_{j+1}^n - U_j^n)}{\Delta x} = 0$$

and prove that U_j^n has an asymptotic property similar to $u(x,t)$.