臺灣大學數學系

九十四學年度博士班入學考試題

數值 PDE

June 3, 2005

1. (a) (10 points) For a finite difference method for solving a PDE numerically, what is meant by the terms "consistency", "stability", and "convergence" ?
   (b) (5 points) How does the Lax Equivalence Theorem relate these terms to each other ?

2. Consider the centered method

$$Q_j^{n+1} = Q_j^n - \frac{a\Delta t}{2\Delta x}\left(Q_{j+1}^n - Q_{j-1}^n\right)$$

for the scalar advection equation $q_t + aq_x = 0$, where $Q_j^n \approx q(x_j, t_n)$ at some point x_j and time t_n, $a \in \mathbb{R} > 0$. Here Δx and Δt are the spatial mesh size and the temporal time step, respectively.

   (a) (10 points) Apply von Neumann analysis to show that this method is unstable in the 2-norm for any fixed $\Delta t/\Delta x$.

   (b) (5 points) Determine the modified equation for the centred scheme. What can you say about the stability of the centered method from the form of the modified equation ?

   (c) (10 points) Give a simple modification of the centred scheme so that the new scheme is stable under suitable condition.

3. Consider the semi-discretized scheme

$$\frac{dQ_j}{dt} = \frac{1}{(\Delta x)^2}\left(\frac{11}{12}Q_{j-1} - \frac{5}{3}Q_j + \frac{1}{2}Q_{j+1} + \frac{1}{3}Q_{j+2} - \frac{1}{12}Q_{j+3}\right), \qquad j \in \mathbb{Z},$$

for the diffusion equation $q_t = q_{xx}$, where $Q_j(t) \approx q(x_j, t)$.

   (a) (5 points) Determine the order of the accuracy of the scheme.

   (b) (5 points) Is the scheme stable ?

4. Suppose that we want to numerically solve the Poisson equation $\nabla^2 q = f$ over a rectangular domain D in two space dimensions with the Dirichlet boundary condition $q|_{\partial D} = g$, for some nontrivial functions f and g.

   (a) (10 points) Describe one approach in detail if we want to go about it by using a 5-point finite difference scheme, *i.e.*, write down the full discretization scheme in matrix form, the order of accuracy of the scheme, discuss a way to solve the resulting linear system of equations, and estimate the amount of work of the whole finite difference scheme

   (b) (5 points) Suppose that we want to use a 9-point finite difference scheme, instead of the 5-point finite difference scheme, what are the advantages and disadvantages if we do so ?

(c) (10 points) Devise a "fast Poisson solver" for that which should perform faster in operation accounts at the least than the method describe in (a) above.

5. (a) (10 points) Explain briefly the basic idea of multigrid methods
   (b) (5 points) Why are some stationary iterative methods such as the Gauss-Seidel method for solving linear systems sometimes called "smoothers" ?
   (c) (5 points) What is meant by "preconditioning"in the conjugate gradient method ?
   (d) (5 points) List at least two types of preconditioners used with the conjugate gradient method.