國立臺灣大學數學系
九十六學年度博士班入學考試試題
科目：離散數學

2007.05.04

(20%) **1.** (a) Show that for an arbitrary sequence $(a_1, a_2, \ldots, a_n)$ of pairwise distinct real numbers there exist indices $i_1, i_2, \ldots, i_k$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $k = \lceil n^{1/2} \rceil$, such that either $a_{i_1} < a_{i_2} < \ldots < a_{i_k}$ or $a_{i_1} > a_{i_2} > \ldots > a_{i_k}$. Show that the bound is not improvable in general, that is, there exists an n-element sequence containing no increasing or decreasing subsequence with more than $\lceil n^{1/2} \rceil$ terms.

(b) Now, consider two sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ of pairwise distinct numbers. Show that indices $i_1, i_2, \ldots, i_k$, $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, always exist with $k = \lceil n^{1/4} \rceil$ such that the subsequences determined by them in both a and b are increasing or decreasing. Show that the bound is not improvable in general.

(20%) **2.** For $n \geq 1$, let a_n be the number of spanning trees of the graph formed from P_n by adding a vertex adjacent to all vertices of $V(P_n)$. For example, $a_1 = 1, a_2 = 3$ and $a_3 = 8$. Determine a_n in term of n.

(20%) **3.** Suppose G is a bipartite graph whose vertex set is partitioned into A and B such that every edge of G has one vertex in A and the other in B. Suppose G has a matching of size A.

(a) Suppose S and T are subsets of A such that $|N(S)| = |S|$ and $|N(T)| = |T|$. prove that $|N(S \cap T)| = |S \cap T|$.

(b) Prove that A has some vertex v such that every edge incident to v belongs to some maximum matching.

(20%) **4.** A *cactus* is a connected graph in which every block is an edge or a cycle. Determine the maximum (resp. minimum) number M (resp. m) of edges in a simple n-vertex cactus. Prove that for any k with $m \leq k \leq M$, there is always a simple n-vertex cactus of k edges.

(20%) **5.** Recall that an *interval graph* is a graph each of whose vertex corresponds to an interval in the real line such that two distinct vertices are adjacent if and only if their corresponding intervals intersect. Prove that $G = (V, E)$ is an interval graph if and only V has an ordering $v_1, v_2, \ldots, v_n$ such that $i < j < k$ and $v_i v_k \in E$ imply $v_j v_k \in E$. Also, prove that interval graphs are perfect.