臺灣大學數學系

九十五學年度博士班入學考試題

科目：離散數學　　2006.6.2

Entering Examination (Ph.D program) for Discrete Mathematics

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1. (20 %) A family of subsets is called an *intersection family* if every two subsets in it intersect. Let $\mathcal{F}$ be an intersection family of subsets of an n-set X. Prove that there is an intersection family $\mathcal{F}'$ of subsets of X such that $\mathcal{F}' \supseteq \mathcal{F}$ and $|\mathcal{F}'| = 2^{n-1}$.

2. (20 %) Let G be the graph whose vertex set is the set of all n-tuples with elements in $\{0, 1\}$, and x is adjacent to y if x and y differ in exactly k positions, where $1 \leq k \leq n$. Determine the number of components of G. Justify your answer.

3. (20 %) Two people play a game on a graph G, alternately choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins. Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy.

4. (20 %) For any ordering $v_1, v_2, \ldots, v_n$ of the vertex set of a graph G, let $\chi(v_1, v_2, \ldots, v_n)$ denote the number of colors needed if a greedy coloring algorithm is applied using the ordering $v_1, v_2, \ldots, v_n$. Define

$$S(G) = \{\chi(v_1, v_2, \ldots, v_n) : v_1, v_2, \ldots, v_n \text{ is an ordering of the vertex set of } G\},$$

$\chi_{\max}(G)$ is the maximum number in $S(G)$ and $\chi_{\min}(G)$ is the minimum. Let $\Delta(G)$ be the maximum degree of a vertex in G.

(1) Prove that $\chi(G) = \chi_{\min}(G) \leq \chi_{\max}(G) \leq \Delta(G) + 1$.

(2) Prove that if G is P_4-free then $\chi_{\min}(G) = \chi_{\max}(G)$.

(3) For any fixed k, is there any tree T such that $\chi_{\max}(G) - \chi_{\min}(G) \geq k$?

5. (20 %) Let S^n denote the set of all n-tuples with entries in the set $S = \{0, 1, *\}$. For any $a, b \in S^n$, let $d_S(a, b) = \sum_{i=1}^{n} d(a_i, b_i)$, where $d(0, 1) = d(1, 0) = 1$ and all other $d(a_i, b_i) = 0$. The *squashed-cube dimension* of a graph G, denoted by $\text{qdim}(G)$, is defined to be the minimum n such that there is a function $f : V(G) \to S^n$ with the property that $d_G(x, y) = d_S(f(x), f(y))$ for any two vertices x and y in G.

(1) Prove that $\text{qdim}(G)$ is finite for any connected graph G.

(2) Determine $\text{qdim}(P_n)$ for any n-path P_n. Justify your answer.

(3) Determine $\text{qdim}(K_{3,3})$ for the complete bipartite graph $K_{3,3}$. Justify your answer.

(4) Determine $\text{qdim}(K_n)$ for any complete graph K_n. Justify your answer.