臺灣大學數學系

九十四學年度博士班入學考試題

離散數學

June, 2005

1. (20 %) Suppose S is a set of size n, where n is a multiple of 8. Find the number of subsets of S with size divisible by 4.

2. (20 %) A family $\mathcal{F}$ of sets is called a *Sperner family* if no member of $\mathcal{F}$ properly contains any other. Let $\mathcal{F}$ be a family of subsets of an n-element set X. Define $b(\mathcal{F})$ to be the family of all subsets Y of X such that (i) $Y \cap F \neq \emptyset$ for all $F \in \mathcal{F}$; (ii) Y is minimal subject to (i) (i.e., no proper subset of Y satisfies (i)).

(a) Prove that $b(\mathcal{F})$ is a Sperner family.

(b) Show that, for any $F \in \mathcal{F}$ and any $y \in F$, there exists $Y \in b(\mathcal{F})$ with $Y \cap F = \{y\}$.

(c) Deduce that $b(b(\mathcal{F})) = \mathcal{F}$.

(d) Let $\mathcal{F}_k$ denote the Sperner family of all k-element subsets of X. Prove that $b(\mathcal{F}_k) = \mathcal{F}_{n+1-k}$ for $k > 0$. What is $b(\mathcal{F}_0)$?

3. (20 %) Let G be a connected graph with n vertices. Define a new graph G' having one vertex for each spanning tree of G, with vertices adjacent in G' if and only if the corresponding trees have exactly $n(G)-2$ common edges. Prove that G' is connected. Determine the diameter of G'.

4. (20 %) An acyclic orientation of a loopless graph is an orientation having no cycle. For each acyclic orientation D of G, let $r(D) = \max_C \lceil a/b \rceil$, where C is a cycle in G and a, b count the edges of C that are forward in D or backward in D, respectively. Fix a vertex $x \in V(G)$, and let W be a walk in G beginning at x. let $g(W) = a - b \cdot r(D)$, where a is the number of steps along W that are forward edges in D and b counts the number that are backward in D. For each $y \in V(G)$, let $g(y)$ be the maximum of $g(W)$ such that W is an x, y-walk (assume that G is connected).

(a) Prove that $g(y)$ is finite and thus well-defined, and use $g(y)$ to obtain a proper $1 + r(D)$-coloring of G. Thus G is $1 + r(D)$-colorable.

(b) Prove that $\chi(G) = \max_D 1 + r(D)$, where D runs over all acyclic orientations of G.

5. (20 %) The *kth power* of a graph $G = (V, E)$ is the graph $G^k = (V, E^k)$, where $E^k = \{uv : 1 \leq d_G(u, v) \leq k\}$. Prove that the cube G^3 of a connected graph G with at least three vertices is Hamiltonian. (Hint: Reduce to the case of trees, and prove it for trees by proving a stronger result that if xy is an edge of the tree T, then T^3 has a Hamiltonian cycle using the edge xy.)