台灣大學數學系

九十三學年度博士班入學考試題

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June 4, 2004

1.

(20 %) Solve the following recurrence relations.

(a)

$$f(n+1) = 2f(n) + f(n-1) - 2f(n-2),\ f(0) = f(1) = 1,\ f(2) = 2.$$

(b)

$$f(n+1) = 1 + \sum_{i=0}^{n} f(i),\ f(0) = 1.$$

2.

Suppose $\mathcal{F}$ is a family of subsets of $\{1, 2, \ldots, n\}$ such that $A \not\subseteq B$ and $B \not\subseteq A$ for any two distinct $A, B \in \mathcal{F}$. Prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. Moreover, prove that if equality holds, then $\mathcal{F}$ consists of all subsets of $\{1, 2, \ldots, n\}$ of size $\lfloor n/2 \rfloor$, or all subsets of size $\lceil n/2 \rceil$ (these are the same if n is even).

3.

(20 %) (a) Prove that every nontrivial tree has at least two maximal independent sets, with equality only for stars. (Note: maximal $\neq$ maximum.) (b) Let T be a tree in which all vertices adjacent to leaves have degree at least three. Prove that T has some pair of leaves with a common neighbors.

4.

(20 %) A *dominating set* in a graph $G = (V, E)$ is a subset D of V such that every vertex in $V - D$ is adjacent to some one vertex in D. The *domination number* $\gamma(G)$ of a graph G is the minimum size of a dominating set in G. A dominating set is *independent* if every two distinct vertices in it are not adjacent.

(a)

Prove that if the diameter of G is at least 3, then $\gamma(\overline{G}) \leq 2$.

(b)

Prove that every claw-free graph, that is G contains no $K_{1,3}$ as an induced subgraph, has an independent dominating set of size $\gamma(G)$.

5.

(20 %) (a) Prove that a graph $G = (V, E)$ is chordal if and only if V has an ordering $v_1, v_2, \ldots, v_n$ such that $i < j < k$, $v_i v_j \in E$, $v_i v_k \in E$ imply $v_j v_k \in E$. (b) Prove that chordal graphs are perfect.