臺灣大學數學系

九十二學年度博士班入學考試題

離散數學

(20 %) The number $f(n)$ of steps required to solve the ``Chinese rings puzzle'' with n rings satisfies $f(1) = 1$ and

$$f(n+1) = \begin{cases} 2f(n), & \text{if } n \text{ is odd,} \\ 2f(n)+1, & \text{if } n \text{ is even.} \end{cases}$$

Prove that $f(n+2) = f(n+1) + 2f(n) + 1$. Hence or otherwise find a formula for $f(n)$ in term of n. 2. (20 %) Suppose $(A_1, A_2, \ldots, A_n)$ is a family of sets satisfying the Hall's condition as follows:

$$|A(J)| \geq |J| \quad \text{for all } J \subseteq \{1, 2, \ldots, n\},$$

where $A(J) = \cup_{j \in J} A_j$. If moreover that each A_i has at least two elements, prove that the family $(A_1, A_2, \ldots, A_n)$ has at least two SDR (system of distinct representatives). 3. (20 %) Prove that if a graph G is P_4-free then either G or its complement $\overline{G}$ is not connected. 4. (20 %) Let G be the **unit-distance graph** in the plane: $V(G) = R^2$, and two points are adjacent if and only if their Euclidean distance is 1. Notice that this is an infinite graph. Prove that $4 \leq \chi(G) \leq 7$. 5. (20 %) The **(Hamiltonian) closure** of an n-vertex graph G, denoted by $C(G)$, is the graph with vertex set $V(G)$ obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains. Notice that the closure of G is well-defined. Prove that for any $n \geq 3$, a simple n -vertex graph is Hamiltonian if and only if its closure is Hamiltonian.