國立臺灣大學數學系

九十七學年度博士班入學考試試題

科目：實分析

2008.04.25

1. (30 pts)

(1) What is (Lebesgue) dominated convergence theorem (DCT)? (5 pts)

(2) What is Fatou's Lemma (FL)? (5 pts)

(3) Is it possible to prove (FL) by (DCT)? Prove your answer (10 pts).

(4) Is it possible to prove (DCT) by (FL)? Prove your answer (10 pts).

2. (30 pts) Let E be a measurable set with finite Lebesgue measure. For $p \geq 1$, define

$$N_p[f] = \left(\frac{1}{|E|} \int_E |f|^p \right)^{1/p}, \quad \forall f \in L^p(E).$$

Can $N_p[f] \leq N_q[f]$ for $p < q$? (10 pts) Can $N_p[f+g] \leq N_p[f] + N_p[g]$ for $f, g \in L^p(E)$? (10 pts) What is the limit $\lim_{p\to\infty} N_p[f] =$? (10 pts) Prove or disprove all your answers.

3. (20 pts)

(1) Can $L^2(\mathbb{R}^n) \subset L^1(\mathbb{R}^n), \forall n \geq 1$? (5 pts) Prove your answer.

(2) Let Ω be a bounded smooth domain in $\mathbb{R}^n, n \geq 2$. Let $\{f_k : k = 1, 2, 3, \cdots\}$ be a bounded sequence in $L^2(\Omega)$. Can the sequence $\{f_k\}$ have a convergent subsequence in $L^1(\Omega)$? Prove or disprove your answer.(15 pts)

4. (20 pts) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^2(\Omega)$ such that as $n \to \infty$, f_n converges to 0 in $L^2(\Omega)$ weakly, to 0 in $L^{3/2}(\Omega)$ strongly, where Ω is a bounded domain in $\mathbb{R}^2$. Can f_n converge to 0 strongly in $L^2(\Omega)$? Prove or disprove your answer.