國立臺灣大學數學系
九十六學年度博士班入學考試試題
科目：實分析

2007.05.04

\# 1. (10 pt) Calculate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ using $\int_0^1 (-x)^k \, dx$.

\# 2. (20 pt) Let $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of measurable functions such that

$$0 \leq f_{n+1} \leq f_n \,, \forall n \in \mathbb{N} \,, \quad \text{and} \quad \lim_{n\to\infty} f_n = f_\infty \quad \text{almost everywhere}\,.$$

Answer the following questions:

(1) Can f_∞ be measurable? (10 pt)

(2) Can $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx = \int_{\mathbb{R}} f_\infty(x)\, dx$? (10 pt)

Prove or disprove your answers.

\# 3. (30 pt) Let $f \in L^1(\mathbb{R})$. For $\xi \in \mathbb{R}$, let $\hat{f}(\xi) = \int_0^\infty e^{-\xi^2 x} f(x)\, dx$. Answer the following questions:

(1) Can $\hat{f} \in L^1(\mathbb{R})$? (15 pt)

(2) Can $\hat{f}$ be differentiable? (15 pt)

Prove or disprove all your answers.

\# 4. (20 pt) Let μ be the standard Lebesgue measure on $\mathbb{R}$. Set Ω as the class of all measurable sets with respect to μ. Assume ν is a finite measure on $\mathbb{R}$. Can there exist a function $f \in L^1(\mathbb{R})$ such that

$$\nu(E) = \int_E f \, d\mu \,, \quad \forall E \in \Omega\, ?$$

Prove or disprove your answer.

\# 5. (20 pt) Let $f \in L^1(\mathbb{R}^n)$ be continuous on $\mathbb{R}^n, n \geq 1$. Can the following limit

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} \frac{\varepsilon}{(\varepsilon^2 + |x|^2)^{(n+1)/2}} f(x) dx$$

exist? If "Yes", find the limit and prove it. If "No", give a counterexample.