臺灣大學數學系

九十四學年度博士班入學考試題

實分析

Jun, 2005

\# 1. (1) Let $\{A_k\}$ be a sequence of nested closed sets in $\mathbb{R}$ such that each A_k is nonempty and $A_{k+1} \subset A_k, \forall k$. Can $\cap_{k=1}^{\infty} A_k$ be nonempty? If yes, prove it. Otherwise, give a counterexample. (10 pt).

(2) Prove that the open interval $(0,1)$ can be represented as a disjoint union of closed intervals. (10 pt)

\# 2. Let $f : [0,1] \to \mathbb{R}$ be an increasing function. Answer and prove or disprove the following questions:

(1) Can f be differentiable almost everywhere on $[0,1]$? (10 pt)

(2) Let f' be the first derivative of f. Suppose f' is Lebesgue integrable on $[0,1]$. Can $\int_0^1 f'\,dx = f(1) - f(0)$? (10 pt)

(3) Can f be Lebesgue integrable? (10 pt)

\# 3. (1) What's the convergence in measure? (5 pt)

(2) Can the convergence almost everywhere imply the convergence in measure? (5 pt)

(3) If the answer of (ii) is yes, prove it. Otherwise, give a counterexample and write the correct statement. (10 pt)

\# 4. Answer and prove or disprove the following questions:

(1) Can the convergence in $L^p([0,1])$ for some $1 < p < \infty$ imply the convergence in $L^1([0,1])$? (10 pt)

(2) Can the convergence in $L^p([0,1]), \forall\, 1 < p < \infty$ imply the convergence in $L^\infty([0,1])$? (10 pt)

\# 5. Can the weak convergence in $L^2([0,1])$ imply the (strong) convergence in $L^2([0,1])$? If the answer is yes, prove it. Otherwise, give an extra condition such that the weak convergence may imply the (strong) convergence in $L^2([0,1])$. Of course, you have to prove all your answers. (10 pt)