台灣大學數學系

九十三學年度博士班入學考試題

實分析

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(1)

Let $Q = (0,1) \times (0,1) = \{(x,y) | 0 < x < 1, \quad 0 < y < 1\}$ be an open square, $f(x,y) > 0$ be a positive measurable function . For almost all x, $\int_0^1 \int (x,y)dy$ exists, and the treated integral $\int_0^1 dx(\int_0^1 fdy)$ exists. Similarly assume $\int_0^1 dy(\int_0^1 f(x,y)dx)$ exists. Can you prove they are the same ?If $g(x,y) \in C^1(Q)$ is a continuously differentiable real-valued function on Q, Can you prove $\int_0^1 \int_0^1 gdydx = \int_0^1 \int_0^1 dydx$? pt

(2)

Sin $\pi z = \pi z(1-z^2)(1-\frac{z^2}{4})(1-\frac{z^2}{9}) \cdots$ If $z = x + io$ has no imaginary part, can you prove this infinite product converges uniformly for $z \in R$? If not, can you take logarithm and differentiate term by term to get cot $\pi z = \frac{1}{\pi^2} + \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{2z}{z^2 - n^2}$?(20/100) pt

(3)

Let $f(x) \in L^2(R)$ be a complex valued square integralle function, $\int_{-\infty}^{\infty} |f|^2 dx < \infty$. Can you prove its Fourier transform $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx$ exists for almost all $\xi \in R$? If yes, is $\widehat{f}(\xi)$ an L^2 function too? (20/100) pt

(4)

Let $\alpha = \frac{1}{2}$ and for each $x \in [-1,1]$ assume $f(x)$ satisfies the hölder condition $|f(x) - f(y)| \leq M|x-y|^{\alpha}$ for all $-1 \leq y \leq 1$ and $M = M(x,f)$ is independent of y. Can you prove there is actually an $H = H(f)$ independent of both x and y so that $|f(x) - f(y)| \leq H \cdot |x-y|^{\alpha}$ for all $(x,y) \in [-1,1] \times [-1,1]$?(20/100) pt

(5)

Let H be a hilbert space and $K \subset H$ a closed subsphce, If $x \notin K$ and d=dist $(X,K) \gneqq 0$

Let $k_1, k_2, \cdots$ be a requence in K so that disk $(x, k_n) \longrightarrow limit = d$ .Can you find a convergent subsequence $k_{i_j} \longrightarrow limit = k_\infty \in K$? If H is a Brunch space instead of hilbert space. can you still find such a convergent subsequence in a closed subspace k ?(20/100)