臺灣大學數學系

九十二學年度博士班入學考試題

分析

A.

Determine which of the following functions is in $L^1(E)$.

$$x^\alpha sin\frac{1}{x}, \quad x \in E = (0,1]; \frac{x^2-y^2}{(x^2+y^2)^2}, \quad (x,y) \in E = (0,1] \times (0,1]$$

where α is a positive constant. Then give an example of a bounded continuous function $f:(0,\infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$, but $f \notin L^p((0,\infty))$ for any $p > 0$.

B.

Denote $I = [0,1]$. Let $I^2 = I \times I \subset \mathbb{R}^2$ be the closed unit square.

$f = f(x,y) : I^2 \to \mathbb{R}$

is a function defined on I^2.

(a)

If $f(x,y)$ is continuous in x for each fixed $y \in I$, and $f(x,y)$ is continuous in y for each fixed $x \in I$, must f be continuous in I^2? Must $f \in L^1(I^2)$?

(b)

If $f(x,y)$ is continuous almost everywhere, and bounded in I^2, prove that the following two Lebesgue iterated integrals are equal:

$$\int_I \int_I f(x,y)dydx = \int_I \int_I f(x,y)dxdy.$$

(c)

If $f(x,y)$ is Lebesgue measurable in x for all fixed $y \in I$, and $f(x,y)$ is continuous
in y for almost all fixed $x \in I$, prove that f is Lebesgue measurable in I^2.

Moreover, must the conclusion be true if we only assume that $f(x,y)$ is Lebesgue measurable with respect to each variable separately?

C.

Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ where p and q are positive constants with

$\frac{1}{p} + \frac{1}{q} = 1.$

Prove that convolution $f * g$ exists, and is continuous in $\mathbb{R}^n$, and lim

$_{|x| \to \infty} f * g(x) = 0.$

Prove aslo that $f * g$ is uniformly continuous in $\mathbb{R}^n$.

D.

Let $I = (a, b)$ be an open interval in $\mathbb{R}$. $f_n : I \to \mathbb{R}(n = 1, 2, 3, \cdots)$ is a sequence of differentiable functions such that lim$_{n \to \infty} \|f_n - f\|_{L^3} = 0$ for some $f : I \to \mathbb{R}$, and $f_n'(x)$ is Cauchy in $L^3(I)$. Prove that f is absolutely continuous in I, and its derivative $f' \in L^3(I)$. Moreover, prove that $f \in L^3(I) \bigcap L^\infty(I)$.