臺灣大學數學系

九十一學年度博士班入學考試題

分析

1.(20 pts) (a) Let $\mathcal{C}, \mathcal{B}, \mathcal{L}$ denote respectively the class of all continuous, Borel measurable, Lebesgue measurable functions on [0, 1]. Prove that $\mathcal{C} \subset \mathcal{B} \subset \mathcal{L}$.

(b) For which class $\mathcal{A}$ of $\mathcal{C}, \mathcal{B}, \mathcal{L}$, the following statement holds:

$f(x) = g(x), a.e. \;\; x$, and $f \in \mathcal{A} \Longrightarrow g \in \mathcal{A}$

Give reason(prove or give a counter-example). pt 2.(20 pts) (a) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on [0, 1]. Prove that

$$\int_0^1 \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \int_0^1 f_n(x)dx.$$

(b) Can $\leq$ in (a) be replaced by $=$? Give reason. pt 3.(20 pts) (a) If $f(x), x \in [0,1]$, is of bounded variation, prove that $f'(x)$ exists $a.e. \;\; x$ and that $x \to f'(x)$ is measurable.

(b) Give an *suitable* upper bound for

$$\int_0^1 f'(x)dx$$

(c) Can the upper bound in (b) become an equality? Give reason. pt 4.(20 pts) (a) Let $f_n \longrightarrow f$ in $L^p[0,1]$, $1 \leq p < \infty$. Let g_n be a sequence of measurable functions such that $|g_n(x)| \leq M < \infty, \quad \forall n \forall x$, and $g_n \longrightarrow g \;\; a.e.$ Prove that $g_n f_n \longrightarrow gf$ in $L^p[0,1]$.

(b) Does (a) hold when $p = \infty$? Give reason. pt 5. Consider

$$I(\lambda) = \int_0^\infty \frac{x^{\lambda-1}}{1+x}dx, \quad 0 < \lambda < 1.$$

(a) Prove that $I(\lambda)$ exists, as a finite positive number.

(b) Evaluate $I(\lambda)$ by the residue method. You need to proceed step by step.