臺灣大學數學系

八十九學年度博士班入學考試題

分析

Choose 4 from the following 6 problems.

1.

(a)

(12) Let f be positive and measurable. Show that

$$\int_0^1 f dx \leq (\int_0^1 \frac{1}{f} dx)^{\frac{1}{2}} (\int_0^1 f^4 dx)^{\frac{3}{8}}$$

(b)

(13) Let f be differentiable and $\frac{d}{dx} f \leq 0$. Show that

$$\int_0^1 x f(x) dx \leq \int_0^1 x dx \int_0^1 f(x) dx$$

2.

Let $f_k(x) = \int_{R^2} \frac{k}{|y|} \frac{1}{(1+|y-kx|^2)^2} dy$ and $c = \int_{R^2} \frac{1}{(1+|y|^2)^2} dy$.

(a)

(12) Show that $\liminf_{k \to \infty} f_k(x) \geq \frac{c}{|x|}$ for $x \neq 0$.

(b)

(13) Show that $\limsup_{k \to \infty} f_k(x) \leq \frac{c}{|x|}$ for $x \neq 0$.

3.

(a)

(13) Assume f is Lipschitz on [0,1], i.e. $\exists L > 0$ such that $| f(x) - f(y) | \leq L | x - y |$, and $f(0) = 0$. Let $g(x) = 0$. Let $g(x) = x^{-\frac{1}{2}} f(x)$. Show that $g'(x) = \frac{d}{dx} g(x)$ exists a.e. on [0,1], $\int_0^x g'(t) dt = g(x)$ and $g(x)$ is absolutely continuous on [0,1].

(b)

(12) Find a Lipschitz function $f(x)$ on [0,1] with $f(0) = 0$ such that $x^{-1} f(x)$ is not absolutely continuous on [0,1].

4.

Let $m(E)$ denote the Lebesgue measure of $E \subset R^n$. $\{f_k\}$ is said to converge in measure on E to f if for every $\varepsilon > 0$,

$$\lim_{k \to \infty} m(\{x \in E : | f_k(x) - f(x) |> \varepsilon\}) = 0 .$$

Assume $f_k, k = 1, 2, \cdots$, and f are measurable and finite on E.

(a)

(12) Show that if $m(E) < \infty$ and $f_k \to f$ a.e. on E, then $f_k \to f$ in measure on E.

(b)

(13) If $f_k \to f$ in measure on E, show that there is a subsequence f_{k_j} such that $f_{k_j} \to f$ a.e. on E.

5.

Let $L^2[0,1]$ denote the real L^2 space on [0,1] with inner product $\langle f.g \rangle = \int_0^1 fg\ dx$.

Let $\{\phi_k\}_{k=1}^{\infty}$ be an orthonormal system in $L^2[0,1]$. Define $c_k(f) = \int_0^1 f\phi_k\ dx$ for $f \in L^2[0,1]$.

(a)

(13) Show that $\sum_{k=1}^{\infty} | c_k(f) |^2 \leq \|f\|_{L^2}^2$.

(b)

(12) Let $\{a_k\}_{k=1}^{\infty}$ be a sequence with $\sum_{k=1}^{\infty} a_k^2 < \infty$. Show that there is a $g \in L^2[0,1]$ such that $a_k = c_k(g)$.

6.

Let m denote the two dimensional Lebesgue measure.

(a)

(12) Let f be a continuous function on [0,1] and $G(f) = \{(x, f(x)) \in R^2 : 0 \leq x \leq 1\}$. Show that $m(G(f)) = 0$.

(b)

(13) Let $\begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases}$, $0 \leq t \leq 1$ be a continuous curve in R^2 and $C = \{(x, y) \in R^2 : x = \phi(t), y = \psi(t), 0 \leq t \leq 1\}$. For a partition $\Gamma = \{0 = t_0 < t_1 < \cdots < t_m = 1\}$, define

$$l(\Gamma) = \sum_{i=1}^{m} \{(\phi(t_i) - \phi(t_{i-1}))^2 + (\psi(t_i) - \psi(t_{i-1})^2)\}^{\frac{1}{2}}.$$

Show that if $\sup_\Gamma l(\Gamma) < \infty$, then $m(C) = 0$.