臺灣大學數學系

八十七學年度博士班入學考試題

分析

There are problems A to D with a total of 100 points. **Please write down your proof or computational steps clearly on the answer sheets.**

A.

Let I be an interval in the real line. $f_n(n = 1, 2, 3, \cdots)$ is a sequence of real-valued functions defined on I. f is also defined on I.

(a)

Define the following terminologies (using $\epsilon - \delta$ definition in (1) and (2)). Each has 4 points.

$$(1). \lim_{n\to\infty} f_n = f \text{ pointwise in } I; \quad (2). \lim_{n\to\infty} f_n = f \text{ uniformly in } I;$$

$$(3). f_n \text{ is uniformly bounded in } I.$$

(b)

Determine which of the following statements is true or false. If true, prove it. Otherwise, give a counterexample. Each has 6 points.

(1)

Suppose $\lim_{n\to\infty} f_n = f$ uniformly in I, and f is continuous in I. Then f_n is also continuous in I for all n large enough.

(2)

Let $\lim_{n\to\infty} f_n = f$ uniformly in I. Suppose $x_n \in I$ is a sequence such that $\lim_{n\to\infty} x_n = a$ with $a \in I$. Then $\lim_{n\to\infty} f_n(x_n) = f(a)$.

(3)

$\lim_{n\to\infty} f_n = f$ uniformly in I iff $\lim_{n\to\infty}(f_n(x_n) - f(x_n)) = 0$ for all sequence x_n of numbers in I.

(4)

Let $\lim_{n\to\infty} f_n = f$ uniformly in I. Then f is bounded in I iff there exists an integer N such that $\{f_n | \, n > N\}$ is uniformly bounded.

(c)

(14 points) Assume that $\lim_{n\to\infty} f_n = f$ pointwise in the bounded interval I. Suppose that each f_n is continuously differentable such that $\int_I |f_n'(x)|^3 dx$ is a

bounded sequence. Prove that f_n is uniformly bounded, and $\lim_{n\to\infty} f_n = f$ uniformly in I.

B.

Define $f(x) = \int_0^\infty \frac{e^{-t}(1-\cos xt)}{t^2} dt$ for real x variable. Find the domain of f. Show that f is twice differentiable in its domain. Find the value of f. (15 points)

C.

Let Γ be an oriented closed curve in the plane without passing through points $(-1,0)$ and $(1,0)$. Compute the line integral

$$\int_\Gamma \frac{2(x^2-y^2-1)dy - 4xydx}{(x^2+y^2-1)+4y^2}.$$

Prove your answer. (15 points)

D.

Let $f(x,y)$ be a continuously differentiable function defined in the disc $D = \{(x,y) | \, x^2+y^2<1\}$. $f(0,0)=0$ and $|f(x,y)| \le 1$ in D.

(a)

(12 points) Prove that there exists a unique continuous function $z=\phi(x,y)$ defined on D such that $|\phi(x,y)| \le 1$ and $z^3 + z(x^2+y^2) = f(x,y)$ on D. Moreover, show that $\phi(x,y)$ is continuously differentiable in D except possibly at $(0,0)$.

(b)

(8 points) When $f(x,y) = (x^2+y^2)^2$, prove that ϕ is continuously differentiable at $(0,0)$. However, when $f(x,y) = (x^2+y^2)^{3/4}$, ϕ is not differentiable at $(0,0)$

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