1. Show that $3^{2n+5} + 160n^2 - 56n - 243$ is divisible by 2^9 for any non-negative integer n. (20%)

2. Let G be a simple group of order 60.

   (a) How many elements of order 5 does G have? (5%)

   (b) How many elements of order 3 does G have? (5%)

   (c) Show that G is isomorphic to A_5. (10%)

3. Let R be a commutative ring with unity and x an indeterminate over k. Let

   $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R[x]$ such that $R[y] = R[x]$. Show that a_1 is

   invertible in R and $a_2, \cdots, a_n$ are all nilpotent. (20%)

4. Let F be an infinite field and x an indeterminate over F. For $a \in F$ let ϕ_a be the

   automorphism of $F(x)$ defined by $f(x) \rightarrow f(x + a)$. Let G be the group of the

   automorphisms of $F(x)$ consisting of all such ϕ_a's. Find the fixed field of G. (20%)

5. Let V be a finite-dimensional vector space over a field F and T a linear transformation

   on V. Show that there exists a positive integer k such that $V = \mathrm{Ker}(T^k) \oplus T^k(V)$.

   (20%)