國立臺灣大學數學系

九十六學年度博士班入學考試試題

科目：代數

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15 (1) Show that the equation

$$(x^2 - 13)(x^2 - 17)(x^2 - 221) \equiv 0(\mod m)$$

is solvable for all $m \geq 2$.

15 (2) Let G be a free abelian group generated by $\{x, y, z\}$. Let H be the subgroup generated by $\{x - y + 2z, -2x + 5y - 4z, 6y + 6z\}$. Determine the group structure of G/H.

20 (3) Show that a group of order p^2q is solvable, where p, q are distinct primes.

15 (4) Consider a dihedral group $D_4 = \{\sigma^i\tau^j | \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma^{-1}\}$ acting on $k[x, y]$ by $\sigma(x) = -y, \sigma(y) = x, \tau(x) = -x, \tau(y) = y$. And moreover, $\sigma(f(x, y)) := f(\sigma(x), \sigma(y)), \tau(f(x, y)) := f(\tau(x), \tau(y))$.

Let $R := \{f(x, y) \in k[x, y] | \sigma(f) = f, \tau(f) = f\}$ be the ring of invariants. Determine R.

20 (5) Determine the Galois group of $x^4 - 5$ over $\mathbb{Q}$, $\mathbb{Q}[\sqrt{5}]$ respectively.

15 (6) Given a homogeneous polynomial $f(x_1, ..., x_n) \in \mathbb{R}[x_1, ..., x_n]$ of degree 2, then $f = y_1^2 + ... + y_t^2 - y_{t+1}^2 - ... - y_{t+s}^2$ by some linear change of variables from $\{x_i\}$ to $\{y_i\}$.