臺灣大學數學系

九十學年度博士班入學考試題

代數

INTRODUCTION:you will need to prove your statements in order to get full credit.Each problem is worth 20 points.

1.

Let k be a field of characteristic 0. Let $f(x_1,\ldots,x_n)$ be a polynomial in the variables $x_1,\ldots,x_n$ such that $f(m_1,\ldots,m_n)=0,\forall(m_1,\ldots,m_n)\in Z_+^n$ (Here Z_+ denotes the set of non-negative intergers.).Prove that f the zero polynomial.

2.

Let V be a vector space over a field k. Suppose that $T:V\to V$ is a diagonalizable linear operator. Let V_λ denotes the eigenspace of V corresponding to eigenvalues λ so that $V=\bigoplus_\lambda V_\lambda$.

a

Let $W\subseteq V$ be a T-invariant subspace,i.e. $T(W)\subseteq W$. Prove that $W=\bigoplus_\lambda(V_\lambda\bigcap W)$.

b

Suppose that $S:V\to V$ is another diagonalizable linear operator such that $ST=TS$. Prove that we can find a basis for V with respect to which both S and T are diagonal.

3.

Let R be a commutative Noetherian ring with identity. Prove that $R[x]$ is Noetherian.

4.

Let $f(x)=x^5-5$

a

Find its spliting field over Q. What is its degree over Q?

b

What can you say about its Galois group? Is it solvable, abelian or even cyclic?

c

Describe explicitly the Galois group?

5.

Let A be a $n\times n$ complete matrix. Let Tr denote the trace of a matrix. Suppose that

$Tr(A) = Tr(A^2) = Tr(A^n) = 0$. Prove that A nilpotent.