臺灣大學數學系

八十八學年度博士班入學考試題

代數

1. Let G be a group which has only a finite number of subgroups. Show that G must be a finite group.
2. Show that a finite group can not be the set-theoretic union of the conjugates of a proper subgroup.
3. Let R be a commutative ring. Suppose that $P_1, \cdots, P_m$ are prime ideals of R and I is an ideal of R such that $I \subseteq \cup_{i=1}^{m} P_i$. Show that $I \subseteq P_i$ for some i.
4. Let R be a ring and $a,\ b$ two nonzero elements in R. Suppose that a is nilpotent (i.e. $a^n = 0$ for some natural number n) and b is a right zero-divisor (i.e. $b'b = 0$ for some $b' \neq 0$). Show that there exists a nonzero element c such that $ac = cb = 0$.
5. Let $f(x)$ be an irreducible polynomial over a field F of characteristic 0. If a and b are two distinct roots of $f(x) = 0$ in some extension over F, show that $a - b \notin F$.
6. For the fields $F_1 = Q(\sqrt{2}\omega)$ and $F_2 = Q(\sqrt{2} + \omega)$, where ω is an imaginary cubic root of unity, determine which of the following holds: (a) $F_1 = F_2$, (b) $F_1 F_2$, (c) $F_2 F_1$, (d) $F_1 \not\subseteq F_2$ and $F_2 \not\subseteq F_1$.