臺灣大學應用數學科學研究所 108 學年度碩士班甄試試題

科目：機率統計

2018.10.19

1. (15%) Specify the joint distribution of R and Θ so that $X = R\cos\Theta$ and $Y = R\sin\Theta$ are independent standard normal random variables.

2. (15%) Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ_0 and variance σ_0^2. Find the mean and variance of $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$, where $\bar{X}$ is the sample mean of $X_1, \ldots, X_n$.

3. (15%) Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ_0^2, where σ_0^2 is an unknown constant. Consider the hypotheses $H_0 : \mu \geq \mu_0$ versus $H_A : \mu < \mu_0$. Compute the power at μ_1 with $\mu_1 < \mu_0$.

4. Let $\{X_{11}, \ldots, X_{1n_1}\}, \cdots, \{X_{k1}, \ldots, X_{kn_k}\}$ be k $(k > 2)$ independent random samples from $N(\mu_{01}, \sigma_0^2), \ldots, N(\mu_{0k}, \sigma_0^2)$, respectively.

(4a) (7%) (8%) Write the corresponding likelihood function and derive the maximum likelihood estimator of $(\mu_{01}, \ldots, \mu_{0k}, \sigma_0^2)$.

(4b) (10%) Consider the hypotheses $H_0 : \mu_{01} = \ldots, \mu_{0k}$ versus $H_A : \mu_{0i} \neq \mu_{0j}$ for some $i \neq j$. Derive the likelihood ratio test with size α, $0 < \alpha < 1$.

5. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from a $Bernoulli(\pi_0)$, $0 < \pi_0 < 1$.

(5a) (10%) Find the uniformly minimum variance unbiased estimator of $\pi_0(1 - \pi_0)$.

(5b) (10%) Suppose that the sample size is large enough. Construct an approximated $(1-\alpha)$, $0 < \alpha < 1$, confidence interval for π_0.

(5c) (10%) Find the smallest sample size to achieve $P(|\widehat{\pi}_n - \pi| \leq e) \approx 1 - \alpha$, where $\widehat{\pi}_n$ is the sample mean.