臺灣大學應用數學科學研究所 104 學年度碩士班甄試試題

科目：機率統計

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1. (15%) Suppose we toss a coin N times independently and let p be the probability of getting head when $N = 1$. Let X and Y denote the number of heads and the number of tails respectively.

   (a) (5%) Prove that X and Y are dependent when $N = 1$.

   (b) (10%) Suppose that the coin is tossed N times. The number of tosses N is a random variable and $N \sim Poisson(\lambda)$. Again, denote the number of heads and tails by X and Y, respectively. Show that X and Y are independent.

2. (20%) Let $X_1, \cdots, X_n \sim Uniform(0,1)$ and $Y_n = \bar{X}_n^2$. Here $\bar{X}_n$ is the average of $X_1, \cdots, X_n$. Find the asymptotic distribution of Y_n. (i.e., Find a_n and b such that $a_n(Y_n - b)$ converges to a non-degenerate distribution.)

3. (20%) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with density $f(\cdot)$. Suppose that $P(X_i > 0) = 1$ and that $\lambda = \lim_{x \to 0} f(x) > 0$. Set $X_{(1)}$ to be $\min\{X_1, \cdots, X_n\}$ and $Y_n = nX_{(1)}$. Determine the asymptotic distribution of Y_n.

4. (20%) Consider the one-sample problem: $Y_i \sim N(\mu, 1)$, $1 \le i \le n$ with the Y_is i.i.d.

   (a) (5%) Determine $\hat{\mu}_c$ which is the maximum likelihood estimator of μ when $|\mu|^2 \le c$. Here $c \ge 0$.

   (b) (7%) Determine the mean square error of $\hat{\mu}_c$. (If you are not sure on your answer obtained in (a), you can assume that $\hat{\mu}_c$ is $\bar{Y}/(1 + c/n)$ where $\bar{Y}$ is the average of Y_i, $1 \le i \le n$.)

   (c) (8%) Determine $\hat{\mu}_{Lasso}$ which is the maximum likelihood estimator of μ when $|\mu| \le c$. Here $c \ge 0$.

5. (25%) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed normally distributed random variables with mean θ and variance 1. Consider testing $H_0 : \theta = 0$ versus $H_a : \theta = \theta_n$. Here $\theta_n > 0$.

   (a) (7%) Determine the rejection region of the most powerful test at level α, $0 < \alpha < 1$. Give reason to justify your answer.

   (b) (8%) Find the power of the test you have in (a) under H_a when $\theta_n = 1/\sqrt{n}$. (i.e. Fnd $\beta(\theta_n)$.) If you are not sure that your answer of (a) is correct, you can answer (b) by assuming that the rejection region is $R = \{n^{-1}\sum_{i=1}^{n} X_i > c_n\}$. You then need to determine c_n.

   (c) (10%) Determine the limit of $\beta(\theta_n)$ with $\theta_n = 1/\sqrt{n}$ as n goes to infinity.