國立臺灣大學應用數學科學研究所114學年度碩士班甄試入學筆試

微分方程與線性代數

1. (15 pts) Solve the following equations.

   (a) $\dfrac{dy}{dx} = \dfrac{y}{x-y}$.

   (b) $xy\dfrac{dy}{dx} + e^y = 0$.

2. Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix}$.

   (a) (15 pts) Find $e^{\mathbf{A}t}$.

   (b) (10 pts) Solve the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

   (c) (10 pts) Let S be an $n \times n$ matrix with real-valued entries. Assume $S^3 = S$ and $0 \le a < 1$. Show that $I_{n\times n} - aS$ is invertible and

   $$(I_{n\times n} - aS)^{-1} = I_{n\times n} + pS + qS^2 \text{ with some } p, q \in \mathbb{R}.$$

3. Let P be the vector space of all polynomials with real coefficients, $Q = \{f \in P : f(0) = f(1) = 0\}$, and $Q_5 = \{f \in Q : \text{degree of } f \le 5\}$. Suppose that

   $$\begin{aligned} T(f) = \quad & x^2 f''(x) + 7xf'(x) + 8f(x), \\ U = \quad & \{f \in Q_5 : \exists g \in Q_5 \text{ such that } T(g) = f\}, \\ V = \quad & \{f \in Q : \int_0^1 f(x)T(g(x))\,dx = 0 \text{ for all } g \in Q\}. \end{aligned}$$

   (a) (10 pts) Find the dimension of U.

   (b) (20 pts) Determine V and find its dimension.

4. Assume that u and v satisfy the equations

   $$\begin{aligned} u'(t) &= \frac{t+1}{u^2(t)+1} - t^2 u(t), \quad u(0) = 1, \\ v'(t) &= \frac{3t+1}{v^2(t)+1} - t^2 v(t), \quad v(0) = 1. \end{aligned}$$

   (a) (10 pts) Show that $u(t) > 0$ for $t \ge 0$.

   (b) (10 pts) Show that $v(t) > u(t)$ for $t > 0$