臺灣大學應用數學科學研究所113學年度碩士班甄試筆試試題

科目：微分方程與線性代數

2023.11.02

1. (25 pts) Solve the following equations.
   (a) $x\dfrac{dy}{dx} + 2y\ln y = 0, x > 0, y > 0$;
   (b) $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{xy}$.

2. (30 pts)
   (a) Solve $\begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$.
   (b) Solve $\begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

3. (25 pts) Let $\mathbf{A}$ be an $n \times n$ matrix with coefficients in $\mathbb{R}$. Assume that the ODE system
$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$
has a solution $\mathbf{z}(t)$ such that $|\mathbf{z}(t)| = 1$ for all $t > 0$.
   (a) Show that $\det \mathbf{A} = 0$ if $\mathbf{A}$ is a symmetric matrix;
   (b) Show that $\mathbf{A}$ has an eigenvalue λ with its real part $Re\lambda = 0$ if $\mathbf{A}$ is not symmetric.

4. (20 pts)
   (a) Let $u(t)$ be a solution of $u'(t) = u(t)(1 - u(t)), u(0) > 0$. Show that $\lim_{t\to\infty} u(t) = 1$.
   (b) Let $v(t)$ be a solution of $v''(t) = -v^3(t), v(0) > 0$. Show that $\lim_{t\to\infty} v(t)$ does not exist.