臺灣大學應用數學科學研究所112學年度碩士班甄試試題

科目：微分方程與線性代數

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1. (30%) Set

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4} & 1 \end{pmatrix},$$

and $X(t) = (x_1(t), x_2(t))^T$.

(a) Calculate $e^{tA} := I + tA + \cdots \frac{t^k}{k!}A^k + \cdots$.

(b) Solve the differential system

$$\begin{cases} x_1'(t) = x_2(t) + e^{3t}, \\ x_2'(t) = -\dfrac{1}{4}x_1(t) + x_2(t) + e^{2t}, \end{cases}$$

with initial condition $X(0) = (2, 0)^T$.

2. (20% points) Solve the differential equation

$$t^2 y''(t) + ty'(t) - y(t) = 0$$

for $t \geq 1$ with $y(1) = 2$ and $y'(1) = 1$.

3. (20 %) State and prove Cayley-Hamilton theorem.

4. (30 %) Let

$$A = \begin{bmatrix} 1 & 2 & 10 & 1 & 1 & 2 \\ 11 & 1 & 2 & 2 & 1 & 4 \\ 1 & 12 & 1 & 1 & 3 & 4 \\ 2 & 1 & 2 & 1 & 14 & 1 \\ 1 & 1 & 2 & 13 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 & 18 \end{bmatrix}$$

Show that

(1) the matrix A is invertible;

(2) A has at least one positive eigenvalue.