# 臺灣大學應用數學科學研究所 105 學年度碩士班甄試試題

## 科目：微分方程與線性代數

2015. 10. 23

1. (25%)
Let $y(t)$ satisfy the third order equation $ty''' + (t+1)y'' + y' = 0$.
(a) Verify that $y(t) = e^{-t}$ is a solution of the equation.
(b) Find the general solution of the equation for $t < 0$.

2. (25%)
Find the general solution of the linear system

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 8 & -1 \\ -2 & -8 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}(t).$$

3. (25%)
Let $\vec{p}, \vec{q}$ and $\vec{r}$ be linearly independent vectors in $\mathbb{R}^3$. Consider the lines $L_1 = \{u\,\vec{p}\,|\,u \in \mathbb{R}\}$ and $L_2 = \{v\,\vec{q} + \vec{r}\,|\,v \in \mathbb{R}\}$.
(a) Show that there is a unique pair $(\vec{x}_0, \vec{y}_0)$ such that $\vec{x}_0 \in L_1$, $\vec{y}_0 \in L_2$, and

$$|\vec{x}_0 - \vec{y}_0| = \inf_{x \in L_1, y \in L_2} |\vec{x} - \vec{y}| > 0.$$

(b) Let $F(u,v) = |v\,\vec{q} + \vec{r} - u\,\vec{p}|^2$ and let $(u(t), v(t))$ be a solution of the system

$$\begin{pmatrix} u'(t) \\ \\ v'(t) \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial F}{\partial u}(u(t), v(t)) \\ \\ -\dfrac{\partial F}{\partial v}(u(t), v(t)) \end{pmatrix}.$$

Show that $\lim_{t\to\infty}(u(t)\,\vec{p}, v(t)\,\vec{q} + \vec{r}) = (\vec{x}_0, \vec{y}_0)$.

4. (25%)
Let

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix}, \; n = 1, 2, 3, \ldots.$$

(a) Show that $x_n = y_n + z_n$ if $(x_0, y_0, z_0) = (1, 1, 0)$.
(b) Determine $(a, b, c)$ such that $ax_n + by_n + cz_n = ax_0 + by_0 + cz_0$, $n = 1, 2, 3, \ldots$ for any given $(x_0, y_0, z_0)$.
(c) Show that $\lim_{n\to\infty} x_n z_n y_n^{-2} = 1$ if $x_0, y_0$ and $z_0$ are positive.