臺灣大學應用數學科學研究所 103 學年度碩士班甄試試題

科目：微分方程與線性代數　　2013. 10. 18

1. (30 points) The solution of the initial-value problem for the system of linear ordinary differential equations of the form

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t), \quad t > 0, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

is given by $x(t) = e^{tA}x_0$, where A is a constant matrix.

   (a) If the matrix A in (1) is defined by

$$A_1 = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix},$$

   what is e^{tA} ? Would $\|x(t)\| \to 0$ as $t \to \infty$?

   (b) Repeat (a) if the matrix A is defined by

$$A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

   (c) Based on the solution of (1), derive the solution of the inital-value problem for the linear differential equation

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + b(t), \quad t > 0, \\ x(0) &= x_0. \end{aligned}$$

2. (30 points) The differential equation for the displacement of a pendulum is

$$\frac{d^2}{dt^2}x(t) = -\frac{g}{l}\sin x(t),$$

where g is the acceleration due to gravity, l is length, and x is the angle of displacement of the pendulum from the vertical position. Assume $g = l = 1$.

Find the critical points of this dynamical system and discuss their stability.

3. (20 points) Let K be a skew-symmetric matrix.

   (a) Show that $Q = (I - K)(I + K)^{-1}$ is an orthogonal matrix.

   (b) Find Q, if K is defined by

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

4. (20 points) Block matrix elimination gives, if the pivot block A is invertible

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

The matrix $D - CA^{-1}B$ is called a *Schur complement.*

   (a) Show that its determinant times $\det A$ equals the determinant of the original block matrix left.

   (b) If $AC = CA$, show that the result in (a) becomes $\det(AD - CB)$.