臺灣大學應數所 102 學年度碩士班甄試試題

科目：微分方程與線性代數　　2012.10.19

ODE and Linear Algebra

1. (20%) Solve the initial value problem

$$\begin{cases} y' = \dfrac{1+3x^2}{3y^2-6y}, \\ y(0)=1 \end{cases}$$

and determine the interval in which the solution is valid.

2. (20%) Determine whether the given vector functions are linearly dependent or independent on the interval $(0,1)$.

a.(10%)

$$e^t\begin{bmatrix}1\\0\\-1\end{bmatrix}, e^t\begin{bmatrix}1\\1\\-1\end{bmatrix}, e^{-t}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

b.(10%)

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}t\\1\\t\end{bmatrix}, \begin{bmatrix}t^2\\0\\t^2\end{bmatrix}$$

3. (20%)

Find constant 4×1 vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ such that the solution of the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & 5 & 3 & -5 \\ 2 & 3 & 2 & -4 \\ 0 & -1 & -2 & 1 \\ 2 & 4 & 2 & -5 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

tends to $(0,0,0,0)^T$ as $t \to \infty$ for any $\mathbf{x}_0 \in S$, where

$$S = \{\mathbf{u} : \mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3, \\ -\infty < a_1, a_2, a_3 < \infty\}.$$

4. (20%) Let $A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, where $a_{ij}=\frac{i}{j}$ for $i,j=1,\cdots,n$ and $n\geq 3$. Find all eigenvalues and eigenvectors.

5. (20%) Let A and B be $n\times n$ symmetric matrices with the same eigenvectors. Prove that $e^{A+B}=e^{A}e^{B}$. Here $e^{A}=\sum_{k=1}^{\infty}\frac{A^{k}}{k!}$.