國立臺灣大學應用數學科學研究所114學年度碩士班甄試入學筆試

微積分

1. (10%) Does there exist a function f such that $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for all $x \in \mathbb{R}$?
    a. (3%) State your method and explain why it works
    b. (7%) Justify your answer

2. (10%) Sketch the graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.
    a. (3%) State your method and explain why it works
    b. (7%) Show the graph

3. (10%) Use Taylor series to approximate $\int_0^1 \sqrt{1+x^{10}}\, dx$ correct to two decimal places.
    a. (3%) State your method and explain why it works
    b. (7%) Show the whole calculation

4. (10%) Prove that

$$\int_0^{\pi/2} sin^{2n}x\, dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{\pi}{2} \quad \text{for } n \in \mathbb{N}$$

    a. (3%) State your method and explain why it works
    b. (7%) Show the whole proof

5. (10%) Evaluate $\int_0^\infty 2e^{-\theta}\sin\theta\, d\theta$
    a. (3%) State your method and explain why it works
    b. (7%) Show the whole calculation

6. (10%) Let $f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$

    Find all continuous points of f
    a. (3%) State your method and explain why it works
    b. (7%) Show the whole calculation

7. (13%) Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.
   a. (3%) State your method and explain why it works
   b. (7%) Show the whole calculation

8. (13%) Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0), (1,0)$ and $(0,1)$. Prove that

$$\iint_R f(x+y)\,dA = \int_0^1 u f(u)\,du$$

   a. (3%) State your method and explain why it works
   b. (7%) Show the whole proof

9. (14%) Evaluate $\iint_S (2x + 2y + z^2)\,dS$, where S is the sphere $x^2 + y^2 + z^2 = 1$.
   a. (3%) State your method and explain why it works
   b. (7%) Show the whole calculation