臺灣大學應用數學科學研究所 112 學年度碩士班甄試試題

科目：微積分

2022.10.20

1. (20%) Let $0 < b < a \leq 1$.
   (a) Show the for $0 < x < 1$, the function $f(x) = \dfrac{1-u^a}{1-u^b}$ is increasing.
   (b) Find $\lim_{x\to 1} f(x)$.

2. (20%) Evaluate the limit or show that it does not exist.
   (a) $\lim_{(x,y)\to(0,0)} \dfrac{xy+y^2}{x^2+2y^2}$.
   (b) $\lim_{y\to 0} \left[\lim_{x\to 0} \dfrac{xy+y^2}{x^2+2y^2} \right]$.
   (c) $\lim_{(x,y)\to(0,0)} \dfrac{x^6}{x^6+(y-x^2)^2}$.

3. (20%) Evaluate the limit.
   (a) $\displaystyle\lim_{n\to\infty} \int_0^1 e^x \cos(nx)\, dx$.
   (b) $\displaystyle\lim_{n\to\infty} \int_0^1 e^x \cos(nx^2)\, dx$.
   (c) $\displaystyle\lim_{n\to\infty} \int_0^1 e^x \cos(nx^n)\, dx$.

4. (20%) Let $f(x)$ and $g(x,y)$ be C^2 functions on $\mathbb{R}$ and $\mathbb{R}^2$ respectively.
   (a) Show that there exists $c \in (-1,1)$ such that $f(1)+f(-1)-2f(0) = f''(c)$.
   (b) Show that there exists $(u,v) \in \mathbb{R}^2$ such that $|u|+|v| < 1$ and
   $$g(1,0)+g(-1,0)-g(0,1)-g(0,-1) = \frac{\partial^2 g}{\partial x^2}(u,v) - \frac{\partial^2 g}{\partial y^2}(u,v).$$

5. (20%)
   (a) Show that $\displaystyle\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}\, dxdydz = \sum_{n=1}^{\infty} \frac{1}{n^3}$.
   (b) Show that $\displaystyle\int_0^1\int_0^1 \frac{1}{1-xy}\, dxdy = \int_0^1\int_0^1\int_0^1 \frac{1}{(1-xyz)^2}\, dxdydz$.
   (c) Evaluate the integral $\displaystyle\int_0^1\int_0^1 \frac{x}{1-xy}\, dxdy$.