臺灣大學應用數學科學研究所 110 學年度碩士班甄試試題

科目：微積分

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1. Suppose that $f(x)$ is differentiable on an open interval I.

   (a) (10 pts) Show that if $f'(a) \neq f'(b)$ for some $a, b \in I$, then for any m between $f'(a)$ and $f'(b)$ there is some c between a and b such that $f'(c) = m$.

   (b) (10 pts) Show that if $f'(x)$ is not continuous at $x = a \in I$ then at least one of the one-sided limits $\lim_{x \to a^+} f'(x)$ and $\lim_{x \to a^-} f'(x)$ does not exist.

2. Suppose that f is a C^2 function defined on $\mathbb{R}$.

   (a) (5 pts) Show that for any $a, b \in \mathbb{R}$ there is some c between a and b such that
$$f(b) - f(a) - f'(a)(b-a) = \frac{f''(c)}{2}(b-a)^2.$$

   (b) (10 pts) Show that there is some number $c \in [n - \frac{1}{2}, n + \frac{1}{2}]$ such that
$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\, dx - f(n) = \frac{f''(c)}{24},$$
   where n is any integer. If $f''(x)$ is decreasing, show that
$$f'\left(n + \frac{3}{2}\right) - f'\left(n + \frac{1}{2}\right) \leq f''(c) \leq f'\left(n - \frac{1}{2}\right) - f'\left(n - \frac{3}{2}\right).$$

   (c) (10 pts) Using the result of (b), give an upper bound and a lower bound for $\sum_{n=k}^{\infty} \frac{1}{n^2}$ and approximate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to within 0.001.

3. f is a continuous function.

   (a) (10 pts) Show that
$$\int_0^x \int_0^y \int_0^z f(t)\, dt\, dz\, dy = \frac{1}{2} \int_0^x (x-t)^2 f(t)\, dt.$$

   (b) (10 pts) Derive a similar formula for $\displaystyle\int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} f(t)\, dt\, dx_n\, dx_{n-1} \cdots dx_2$, $n > 3$.

4. (10 pts) Let $f(x,y,z)$, $g_1(x,y,z)$, and $g_2(x,y,z)$ be C^1 functions. Suppose that $\vec{\nabla} g_1(x,y,z)$ and $\vec{\nabla} g_2(x,y,z)$ are linearly independent. Consider the problem of maximizing $f(x,y,z)$ subject to the constraints $g_1(x,y,z) = a_1$ and $g_2(x,y,z) = a_2$, where a_1, a_2 are constants. Let $(x(a_1,a_2), y(a_1,a_2), z(a_1,a_2))$ be the point where the maximum value is obtained with corresponding Lagrange multipliers $\mu_1(a_1,a_2), \mu_2(a_1,a_2)$ which means that at $(x(a_1,a_2), y(a_1,a_2), z(a_1,a_2))$, $\vec{\nabla} f = \mu_1 \vec{\nabla} g_1 + \mu_2 \vec{\nabla} g_2$. Suppose that $x(a_1,a_2)$, $y(a_1,a_2)$, and $z(a_1,a_2)$ are differentiable functions of a_1, a_2. Compute
$$\frac{\partial}{\partial a_j} f(x(a_1,a_2), y(a_1,a_2), z(a_1,a_2)), \quad j = 1, 2.$$
   Write your answers in terms of $\mu_1(a_1,a_2)$ and $\mu_2(a_1,a_2)$.

5. (a) (15 pts) Find the average value of $f(x,y,z) = x^2 - y^2 + 3z^2$ on the sphere $S_r = \{(x,y,z) | (x-1)^2 + y^2 + z^2 = r^2\}$ where $r > 0$ is a constant.

   (b) (10 pts) Suppose that $u(x,y,z)$ is harmonic on an open region D which contains a ball $B_{\vec{a}}(r) = \{\vec{x}\ | |\vec{x} - \vec{a}| \leq r\}$. Find the average value of u on the sphere $S_{\vec{a}}(r) = \{\vec{x}\ | |\vec{x} - \vec{a}| = r\}$.