臺灣大學應用數學科學研究所 108 學年度碩士班甄試試題

科目：微積分

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1. (10 pts) Compute the following integral

$$\int_1^2 \int_{x^2}^{8-x^2} \left[8x \exp((y+x^2)^2 - 4(y+x^2))\right] dydx.$$

2. (15 pts) Suppose $f(x)$ is a differentiable function from the reals into the reals. Suppose $f'(x) > f(x)$ for all $x \in R$, and $f(x_0) = 0$. Can we conclude that $f(x) > 0$ for all $x > x_0$? Justify your answer and state clearly on the theorems you used.

3. (20 pts) Suppose we know the following facts about the functions g and h:

   (a) $g(x) > 0$ on $x \in (0, 1/2)$ and $(3/2, 2)$, $g(x) < 0$ on $x \in (1/2, 3/2)$, and $g(1/2) = g(3/2) = 0$.

   (b) $h(x) > 0$ on $(0, 1)$, $h(x) < 0$ on $(1, 2)$, and $h(0) = h(1) = h(2) = 0$.

   (c) $g'(x)h(x) + g(x)h'(x) = g(2x)$.

   (d) $g(x+2) = g(x)$.

   (e) $\int_0^{1/2} g(x)h(x)dx = 2$.

   (f) $\int_c^{c+1} g(x)h(x)dx = 0$ for all $c \in R$.

   Use this information to sketch the graph of f on the interval $[0, 2]$, where

$$f(x) = \int_0^x g(t)h(t)dt.$$

   In your answer, provide reasons to justify your plot.

4. (15 pts) Let $a \in R$.

   (a) (5 pts) Show that if $\{a_n\}$ is a sequence in R, which converges to a, then

$$\lim_{n\to\infty} \left(1 + \frac{a_n}{n}\right)^n = \exp(a).$$

   (b) (10 pts) Let f be a real-valued differentiable function on R such that the second derivative $f''(0)$ at the origin exists, with $f(0) = f'(0) = 0$ and $f''(0) = a$. Determine the limit of $[f(x/\sqrt{n})]^n$ for each real $x \in R$.

5. (15 pts) Determine the set of all values of α or which the function

$$f(x,y) = \begin{cases} \frac{|x|^{3\alpha} + |y|^{7-\alpha}}{x^2+y^2}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0, \end{cases}$$

   is differentiable at $(0, 0)$. Justify your answer.

6. (10 pts) Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both be convergent series of positive terms. For each of the following decide whether the given series must always converge, must always diverge or if it is impossible to tell. If the series always converges or always diverges, give reasons. If it is impossible to tell, give an explicit example of series a_k and b_k both convergent, for which the series in question diverges and an explicit example of series $\sum a_k$ and $\sum b_k$ both convergent, for which the series in question converges. (5 points each.)

   (a) (5 pts) $\sum_{k=1}^{\infty}(a_k/b_k)$.

   (b) (5 pts) $\sum_{k=1}^{\infty} \ln(a_k b_k)$.

7. (15 pts) Consider the following constrained minimization problem.

$$\min_{x,y} x^2+3xy+5y^2+0.5x \quad \text{under the constraint } 3x+2y+2 \leq 0 \text{ and } 15x-3y-1 \leq 0.$$