臺灣大學應用數學科學研究所 107 學年度碩士班甄試試題

科目：微積分

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1. (a.) Let $f : (a, b) \to \mathbb{R}$. Suppose $f'' > 0$ on (a, b). Show that the graph $y = f(x)$ is above all its tangent lines. (10%). (b.) Suppose $g(x)$ is differentiable on an open interval $(-a, a)$. Determine wether or not there exists a small neighborhood of 0 on which the tangent line of the graph $y = g(x)$ at $(0, g(0))$ only intersects the graph once. Prove your answer or give a counterexample. (10%)

2. Let $\phi(x)$ be a continuous function on $[0, 2]$ that $\phi(0) = \phi(2) = 0$. Suppose that ϕ' is continuous and

$$|\phi'(x)| \leq M$$

on $(0, 2)$ for some $M > 0$. Show that there exists a constant $C > 0$ such that

(a.) $$\int_0^2 \phi(x)\sin(kx)dx \leq C\frac{1}{|k|}, \qquad (5\%)$$

(b.) $$\int_0^2 \phi(x)\sin\left(k(x^2 - 2x + 1)\right)dx \leq C\frac{1}{\sqrt{|k|}}, \qquad (10\%)$$

(c.) $$\int_0^2 \phi(x)\sin\left(k(x^2 + 2x + 1)\right)dx \leq C\frac{1}{|k|}. \qquad (5\%)$$

3. Let $\partial B(0, 1)$ be the unit sphere in $\mathbb{R}^3$ centered at the origin and x be a point in $\mathbb{R}^3$. Evaluate the surface integral

$$\int_{\partial B(0,1)} \frac{1}{|x - y|} dS(y)$$

for the cases $0 \leq |x| < 1$ and $1 < |x|$. (20%)

4. (a.)Determine the radius of convergence of the following power series

$$x + 2x^2 + 3x^3 + 4x^4 \ldots + nx^n \ldots. \qquad (10\%)$$

(b.) Evaluate

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n. \qquad (10\%)$$

5. Let F be a vector field in $\mathbb{R}^3$:

$$F(x, y, z) = \frac{1}{r^3}(x, y, z),$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Let

$$S_1 = \{(x, y, z) | 5x^2 + 6y^2 + 7z^2 = 10000\},$$
$$S_2 = \{(x, y, z) | (x - 11)^2 + y^2 + 2z^2 = 1\}.$$

Evaluate

(q.) $$\int_{S_1} F \cdot n(\zeta)dS(\zeta), \qquad (10\%)$$

(b.) $$\int_{S_2} F(\zeta) \cdot n(\zeta)dS(\zeta), \qquad (10\%)$$

where $n(\zeta)$ is the outer unit normal vector at ζ and " $\cdot$ " represents the inner product.