臺灣大學應用數學科學研究所 106 學年度碩士班甄試試題

科目：微積分

2016. 10. 21

(1) (20 pts) Let $C = \int_{-\infty}^{\infty} \exp(-x^2)dx$ and let S_p be the $(p-1)$-dimensional *surface area* of the unit sphere in R^p (so $S_2 = 2\pi$, $S_3 = 4\pi/3$).

(a) (8 pts) Prove that $C^p = S_p\Gamma(p/2)/2$, where $\Gamma(p) = \int_0^{\infty} t^{p-1}\exp(-t)dt$. (Hint: Evaluate the integral of $\exp(-(x_1^2 + \cdots + x_p^2))$ over R^p in rectangular and polar coordinates.)

(b) (4 pts) Show that $s\Gamma(s) = \Gamma(s+1), \Gamma(1) = 1$.

(c) (8 pts) Evaluate C. (Hint: $S_2 = 2\pi$.)

(d) (4 pts) Evaluate S_4.

(2) (20 pts) Show that

$$\int_0^{\infty} \frac{t\exp(-t/2)}{1-\exp(-t)} = 4\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Please give reasons for your derivation.

(3) (20 pts) Consider the seris $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$.

(a) (14 pts) Show that the series converges for every x.

(b) (6 pts) Provide an argument that the sum is a continuous function on $(-\infty, \infty)$.

(4) (20%) Consider a map f from R^3 to R^3 where

$$f(x,y,z) = \left(\frac{\sin y}{4}, \frac{\sin z}{3} + 1, \frac{\sin x}{5} + 2\right).$$

(4a) (10%) Find the constant c such that

$$d(f(x,y,z), f(x',y',z')) \le cd((x,y,z),(x',y',z')).$$

Here $d((x,y,z),(x',y',z')) = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$.

(4b) (5%) Show that there exists a unique fixed point.

(4c) (5%) Give the number of iteration to the fixed point within 0.001.

(5) (20 pts) (a) (8 pts) Use Lagrange multipliers to show that $f(x,y,z) = z^2$ has only one critical point on the surface $x^2 + y^2 - z = 0$.

(b) (6 pts) Show that the one critical point is a minimum.

(c) (6 pts) Sketch the surface. Why did Lagrange multipliers not find a maximum of f on the surface?