臺灣大學應用數學科學研究所105學年度碩士班甄試試題
科目：微積分

2015.10.23

(1) (20 pts)

(a) (8 pts) Evaluate the four-dimensional volume of the unit ball $x^2 + y^2 + z^2 + v^2 = 1$ in R^4.

(b) (12 pts) Evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-2Q(x,y,z))dxdydz.$$

Here $Q(x,y,z) = x^2 + y^2 + z^2 + xy + yz$.

(2) (20 pts) Use the fact that there is a real number e with the property that

$$0 < n!\left(e - \sum_{r=0}^{n} \frac{1}{r!}\right) < \frac{1}{n}$$

for all integers $n \geq 1$. Assuming only this property of e, prove that e is irrational.
Hint: Assume on the contrary that e = p/q; choose a suitable value for n and deduce that there would be an integer k such that $0 < k < 1$.

(3) (20 pts) Consider the seris $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$.

(a) (14 pts) Show that the series converges for every x.

(b) (6 pts) Provide an argument that the sum is a continuous function on $(-\infty, \infty)$.

(4) (20 pts) Let s_{jk} be non-negative numbers, $1 \leq j, k < \infty$. Suppose that

$$\sum_{k=1}^{\infty} s_{jk} = 1 \quad \text{for all } j$$

and

$$\lim_{j\to\infty} s_{jk} = 0 \quad \text{for all } k.$$

Let $\{x_j\}$ be a convergent sequence of real numbers. Define

$$y_j = \sum_{k=1}^{\infty} s_{jk}x_k.$$

Prove that the sequence $\{y_j\}$ converges and

$$\lim_{j\to\infty} y_j = \lim_{k\to\infty} x_k.$$

(5) (20%) Consider a sequence $a_1, \cdots, a_n, \cdots$ with $a_1 = 1$ and $a_{n+1} = 1 + 1/a_n$.

(5a) (5%) Assume that $\lim_{n\to\infty} a_n$ exists and determine its limit τ.

(5b) (10%) Prove that $|a_{k+1} - \tau| \leq \tau^{-k}$.

(5c) (5%) Prove (from the definition) that $a_n \to \tau$ as $n \to \infty$.