臺灣大學應用數學科學研究所 104 學年度碩士班甄試試題

科目：微積分

2014.10.24

1. (20%) Let $f(x)$ be a convex function on (a,b), i.e., for any $a<x<y<b$ and $0\le\theta\le1$,

$$f((1-\theta)x+\theta y)\le(1-\theta)f(x)+\theta f(y).$$

Show that for any $c\in(a,b)$, both the righthand and the lefthand derivatives,

$$\lim_{x\to c+}\frac{f(x)-f(c)}{x-c}\quad\text{and}\quad\lim_{x\to c-}\frac{f(x)-f(c)}{x-c},$$

exist. From this, argue that a convex function f on (a,b) must be continuous. *Hint*: show that for $a<s<t<u<b$, we have

$$\frac{f(t)-f(s)}{t-s}\le\frac{f(u)-f(s)}{u-s}\le\frac{f(u)-f(t)}{u-t}.$$

2.(20%) Determine the line integral $\int_L \mathbf{F}\cdot\mathbf{t}ds$, where

$$\mathbf{F}=(e^{-y}-ze^{-x})\mathbf{i}+(e^{-z}-xe^{-y})\mathbf{j}+(e^{-x}-ye^{-z})\mathbf{k}$$

and L is the path described by

$$(x,y,z)=\left(\frac{\ln(1+t)}{\ln 2},\sin(\frac{\pi t}{2}),\frac{1-e^t}{1-e}\right).$$

for $0\le t\le 1$.

3. (20%) Show that the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}$ converges and find its value.

4. (20%) Let f be a real-valued function on $(-1,1)$. Assume that f is continuous at 0, $f(0)=0$, and $f(x)\neq0$ for $x\neq0$. Does the limit

$$\lim_{x\to0}\frac{\ln(1+\sin f(x))}{f(x)}$$

exist? If it does, find its limit. Please justify your answer.

5. (20%) (a) Find all extrema of $f(\mathbf{x})=\sum_{k=1}^n x_k^2$ subject to the constraint $\sum_{k=1}^n|x_k|^p=1$, where $p>1$.
(b) Prove that there exist constants a_n, b_n, depending on n, such that for any real vector $\mathbf{x}=(x_1,x_2,\cdots,x_n)$

$$a_n\left(\sum_{k=1}^n|x_k|^p\right)^{1/p}\le\left(\sum_{k=1}^n x_k^2\right)^{1/2}\le b_n\left(\sum_{k=1}^n|x_k|^p\right)^{1/p},$$

where $1\le p\le2$. Find optimal a_n and b_n.