臺灣大學數學系

九十九學年度碩士班甄試試題

科目：機率統計 2009.10.30

1. (20 pts) Suppose that (X, Y) is uniformly distributed over the region

$$\{(x, y) : 0 < |y| < x < 1\}.$$

(1a) (5 pts) Find the joint density of (X, Y).

(1b) (5 pts) Find the marginal densities $f_X(x)$.

(1c) (5 pts) Are X and Y independent? Please state reason to support your answer.

(1d) (5 pts) Find $E(X|Y)$ and $E(Y|X)$.

2. (20 pts) Let $T_1, \ldots, T_n$ be a random sample from an exponential distribution with mean 1. (i.e. Its density function is $\exp(-x)$ over $x \in [0, \infty)$. Denote its order statistics by $T_{(1)} < \cdots < T_{(n)}$.

(2a) (15 pts) Determine a_n such that, for any positive x,

$$\lim_{n \to \infty} P(T_{(n)} \le a_n x) \text{ converges to a limit which is greater than 0 and less than 1.}$$

(2b) (5 pts) Determine the asymptotic distribution of $T_{(n)}$ as n goes to infinity.

3. (20 pts) Suppose $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} Poisson(\lambda)$.

(3a) (8 pts) How would you construct a confidence interval for λ when n is large? (You need to specify which table you need to check for getting a 95% confidence interval.)

(3b) (8 pts) What is a more accurate interval using variance stabilizing transforms?

(3c) (4 pts) If a preliminary estimate suggests that $\lambda \approx 1$, how large a sample should you collect to ensure that the variance stabilized confidence interval has length ≤ 0.1?

4. (20 pts) Suppose $X_1, \ldots, X_n$ are independent random variables, and for each i, X_i follows the exponential distribution with $E(X_i) = i\beta$, where β is an unknown parameter. (i.e., The density function of X_i is $(i\beta)^{-1}\exp(-x/(i\beta))$.)

(4a) (8 pts) Compute the maximum likelihood estimate of β, $\hat{\beta}_n$.

(4b) (7 pts) Determine $E(\hat{\beta}_n)$ and $Var(\hat{\beta}_n)$

(4c) (5 pts) Show that $\hat{\beta}_n$ is a consistent estimator of β.

5. (20 pts) We wish to test the null hypothesis H_0 that a given die is fair ($p_1 = \cdots = p_6 = 1/6$) against the alternative H_a, that the die is biased in the following way:

$$p_1 = 1/5, p_6 = 1/10, p_2 = p_3 = p_4 = p_5 = 7/40.$$

It is rolled 10 times.

(5a) (6 pts) Describe the distribution of (n_1, n_6) under both H_0 and H_a, where n_1 (respectively n_6) is the number of 1s (respectively 6s) among the 10 rolls.

We now plan to carry out a test of H_0 with alternative H_a, on the basis of the outcome of the 10 rolls. Three test statistics are under consideration to test this null hypothesis: the number n_1 of 1s; the number n_6 of 6s; and the likelihood ratio statistic.

(5b) (9 pts) For each of these three statistics, determine a cut off defining a rejection region has a Type 1 error as close to 5% as is achievable. (You don't need to give numerical result. Instead, state clearly what you will do when you have certain table, certain approximation, or calculator.)

(5c) (5 pts) Using the cut offs determined in (5b), calculate the power of one test you choose, that is, the probability of rejecting the null hypothesis H_0 when the alternative H_1 is true. (You don't need to give numerical result. Instead, state clearly what you will do when you have certain table, certain approximation, or calculator.)