臺灣大學數學系
九十六學年度碩士班甄試試題
科目：機率統計

2006.11.3

1. If X and Y are independent random variables with the same density $f(x) = \lambda^{-1}e^{-x/\lambda}\mathrm{I}_{(0,\infty)}(x)$, where $\lambda > 0$, $\mathrm{I}_{(0,\infty)}(x)$ equals 1 if $x \in (0,\infty)$ and 0 otherwise. Let $Z = X + Y$.

   (a) (6%) Find the joint density function of X and Z.

   (b) (6%) Find the conditional distribution of Z given $X = x$ for any $x > 0$.

   (c) (8%) Find and compare the conditional variance of Z given $X = x$, $x > 0$, and the variance of Z.

2. Suppose that $X_1, ..., X_n$ is a random sample from a continuous distribution function F. Let $X_{(1)}, ..., X_{(n)}$ be the order statistics of the sample.

   (a) (10%) Find the distribution of the random variable $F(X_{(k)})$, for any $k = 1, .., n$.

   (b) (10%) Find the limiting distribution of $n\{1 - F(X_{(n)})\}$ when the sample size n is large.

3. A manufacturer wants to estimate the proportion $p, 0 < p < 1$, of defective items the company produces. Since they rarely produce defective items, n workers are asked to continue inspecting until they each has observed one defective item. Let X_i be the number of items inspected by the i-th workers, $i = 1, \ldots, n$.

   (a) (6%) What assumptions are need so that the Geometric model $X_1, ..., X_n$ i.i.d. with probability density function $f(x) = p(1-p)^{x-1}, x = 1, 2, ...$ is reasonable?

   (b) (8%) Find the maximum likelihood estimator of p, denoted as $\hat{p}_1$.

   (c) (10%) Let $\hat{p}_2 = (n-1)\left(\sum_{i=1}^{n} X_i - 1\right)^{-1}$. Show that $\hat{p}_2$ is the uniformly minimum variance unbiased estimator of p.

   (d) (10%) Compare the asymptotic mean squared errors of $\hat{p}_1$ and $\hat{p}_2$.

4. Suppose that $X_1, ..., X_n$ is a random sample from a Gamma distribution with density $f(x) = \Gamma(\nu)^{-1}\theta^{-\nu}x^{\nu-1}e^{-x/\theta}\mathrm{I}_{(0,\infty)}(x)$, where $\nu > 0$ is some known constant and $\theta > 0$ is an unknown parameter.

   (a) (12%) Construct a uniformly most powerful test, with significance level α, for testing the hypotheses H_0: $\theta \in \{0.5, 1, 1.6, 1.7, 2\}$ against H_1: $\theta \in \{2.5, 3, 6, 8, 10\}$.

   (b) (6%) Suppose that $\nu = 0.2$ and there is a sample of size 180 with sample mean 0.3. Under significance level $\alpha = 0.05$, does the test in (a) reject H_0 or not? You may like to use an appropriate normal approximation, and the 95-th percentile of the standard normal distribution is 1.645.

   (c) (8%) For the test in (a), derive an appropriate approximation to the p-value of any given observed value of the sample mean $n^{-1}\sum_{i=1}^{n} X_i$.