臺灣大學數學系

九十五學年度碩士班甄試入學試題

機率與統計

Nov, 2005

1. (5%)(10%) Let $X_1, \cdots, X_n$ be a random sample from a finite population $\{X^{(1)}, \cdots, X^{(N)}\}$. Show that $E[\bar{X}] = \mu$ and $E[\frac{N-1}{N}S^2] = \sigma^2$, where $\bar{X} = \frac{1}{n}\sum_{i=1} nX_i$, $\mu = \frac{1}{N}\sum_{i=1}^{N} X^{(i)}$, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$, and $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(X^{(i)} - \mu)^2$.

2. (10%)(10%) Consider the hierarchical model $X_n|W_n = \omega_n \sim N(0, \omega_n + (1-\omega_n)\sigma_n^2)$, $W_n \sim Bernoulli(p_n)$. Show that $X_n \to N(0,1)$ and $Var(X_n) \to \infty$ as $p_n \to 1$, $\sigma_n \to \infty$ and $(1-p_n)\sigma_n \to \infty$.

3. Let $X_1, \cdots, X_n$ be a random sample from $Exponential(1)$ and $Z \sim N(0,1)$.

   (3a) (10%) Show that $\lim_{n\to\infty} P(\sqrt{n}(\bar{X}_n - 1) \leq z) = P(Z \leq z)$ for all $z \in R$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (3b) (10%) From (3b), show that $\frac{\sqrt{n}}{\Gamma(n)}(z\sqrt{n} + n)^{n-1}\exp(-z\sqrt{n} - n) \approx \frac{1}{\sqrt{2\pi}}\exp(-z^2/2)$.

4. (7%)(8%) Let $X_1, \cdots, X_n$ be a random sample from $Poisson(\lambda)$, and let $\bar{X} = \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Show that $E[S^2|\bar{X}] = \bar{X}$ and $Var(S^2) > Var(\bar{X})$.

5. Let $X_1, \cdots, X_n$ be a random sample from a one parameter exponential family $f(x|\theta) = \exp(\theta h(x) - H(\theta)g(x))$, where $H'(\theta) = h(\theta)$ and $h'(\theta) > 0$.

   (5a) (2%)(3%) Show that $E[h(X)|\theta] = h(\theta)$ and $Var(h(X)|\theta) = h'(\theta)$.

   (5b) (10%) Find the uniformly most powerful level α test of $H_0 : \theta \leq \theta_0$ versus $H_A : \theta > \theta_0$.

6. (10%) Consider the multiple regression models $Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \epsilon_i$ and $Y_i - \bar{Y} = \beta_0^* + \beta_1^*(X_{i1} - \bar{X}_1) + \cdots + \beta_p^*(X_{ip} - \bar{X}_p) + \epsilon_i^*$ with $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$, $\bar{X}_j = \frac{1}{n}\sum_{i=1}^n X_{ij}$, $i = 1, \cdots, n$, $j = 1, \cdots, p$. Here, both ϵ_i's and ϵ_i^*'s are assumed to be uncorrelated with $\epsilon_i \sim (0, \sigma^2)$ and $\epsilon_i^* \sim (0, \sigma^{*2})$. Show that the least squares estimators $(\hat{\beta}_1, \cdots, \hat{\beta}_p)$ of $(\beta_1, \cdots, \beta_p)$ are same with the least squares estimators $(\hat{\beta}_1^*, \cdots, \hat{\beta}_p^*)$ of $(\beta_1^*, \cdots, \beta_p^*)$.