臺灣大學數學系

九十二學年度碩士班甄試入學試題

機率統計

Nov 29, 2002

1.

Let $(\Omega, \mathcal{B}, P)$ be a probability space and $A,\ B \in \mathcal{B}$ with $P(A) > 0$ and $P(B) > 0$. Let I_A and I_B be the indicator functions of A and B respectively. Define $\rho(A, B) =$ the correlation between I_A and I_B.

(a)

(10%) Express $\rho(A, B)$ in terms of $P(A)$, $P(B)$ and $P(A \cap B)$.

(b)

(10%) Show that

$$\rho(A, B) > 0 \Leftrightarrow P(A|B) > P(A) \Leftrightarrow P(B|A) > P(B)\,,$$

and

$$\rho(A, B) < 0 \Leftrightarrow P(A|B) < P(A) \Leftrightarrow P(B|A) < P(B)\,.$$

(c)

(5%) When will $\rho(A, B) = -1$? When will $\rho(A, B) = 1$?

2.

Suppose a bivariate random vector (X, Y) has joint pdf

$$f(x, y) = b\,(c^2 + x^2 + y^2)^{-3/2},\ -\infty < x < \infty, -\infty < y < \infty\,,$$

where $c > 0$ is a constant.

(a)

(5%) Find the value of b.

(b)

(10%) Find the marginal pdf's of X and Y. Does EY exist?

(c)

(10%) Find the conditional pdf of Y given $X = x$. Does the conditional expectation $E(Y|X = x)$ exist?

3.

Let $X_1, \cdots, X_n$ be a random sample from a distribution whose first four moments exist. and let

$$S_n^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1).$$

(a)

(8%) Show that $E(S_n^2) = \sigma^2$, where $\sigma^2 = Var(X_1)$.

(b)

(6%) Show that $Var(S_n^2)$ exists and tends to zero as $n \to \infty$.

(c)

(6%) Show that S_n^2 converges in probability to σ^2 as $n \to \infty$.

4.

Let $X_1, \cdots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution where $\mu \in [\mu_0, \infty)$ for some known μ_0.

(a)

(10%) Find the MLE for μ.

(b)

(10%) For a given level of significance $\alpha < 0.5$, construct the likelihood ratio test for $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$.

(c)

(10%) For a given level of significance $\alpha < 0.5$, construct the Uniformly Most Powerful test for $H_0 : \mu = \mu_0$ versus $H_1 : \mu > \mu_0$.