臺灣大學數學系
九十七學年度碩士班甄試試題
科目：數值分析(含程式設計)　　2007.11.02

1. (20 points) Devise a numerical approximation to $u''(x_2)$ based on data values U_1, U_2, and U_3, at three unequally spaced points x_1, x_2, and x_3, and compute the error in this approximation.

2. (20 points) Devise a numerical algorithm to find the maximum of the following integral

$$I(\alpha) = \int_0^2 (2 + \sin(10\alpha))x^\alpha \sin\left(\frac{\alpha}{2-x}\right) dx$$

when $\alpha \in [0, 5]$.

3. (20 points) Consider a rank-1 matrix of the form $P = uu^T$, where $u \in \mathbb{R}^m$ is a unit vector.
   (a) Show that P is an orthogonal projector.
   (b) With $u = (1,1)^T/\sqrt{2}$, what spaces do P and $I - P$ project onto ?
   (c) If we now have n linearly independent vectors $a_1, a_2, \cdots, a_n$ in $\mathbb{R}^m$, $m > n$, what is the orthogonal projector that project any nontrivial vector onto the range of $A = [a_1, a_2, \cdots, a_n]$?

4. (20 points) Suppose $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix. Describe the Cholesky factorization in the algorithmic details that yields $A = R^TR$, where R is upper-triangular.

5. (20 points) The *Gerschgorin's theorem* which holds for any $m \times m$ matrix A, symmetric or nonsymmetric, states that every eigenvalue of A lies in at least one of the m circular disks in the complex plane with centers a_{ii} and radii $\sum_{j\neq i}|a_{ij}|$. Moreover, if n of these disks form a connected domain that is disjoint from the other $m - n$ disks, then there are precisely n eigenvalues of A within this domain.

   Give estimates based on *Gerschgorin's theorem* for the eigenvalues of

$$A = \begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}, \qquad |\epsilon| \leq 1.$$