臺灣大學數學系
101學年度碩士班甄試試題
科目：數值分析(含程式設計)

2011.10.21

1. (20 points) Steffensen's method for root-finding of scalar nonlinear equations $f(x) = 0$ takes the form

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \qquad n \geq 1.$$

Prove that the method is second-order accurate and is convergent provided that $f'(\alpha) \neq 0$ at the root $f(\alpha) = 0$ and $f \in C^2$ near the root.

2. (35 points) Let $A \in \mathbb{R}^{m \times m}$ be nonsingular.

   (a) Show that $A$ has a unique $LU$ factorization if and only if for each $k$ with $1 \leq k \leq m$, the upper-left $k \times k$ block $A_{1:k,1:k}$ is nonsingular, where $L$ is a unit lower triangular matrix and $U$ is an upper triangular matix.

   (b) Describe the basic steps in pseudo-code form for $LU$ factorization of $A$.

3. (30 points) Let $\Pi_n$ denote the vector space of all polynomials of degree at most $n$. Suppose now we want to find a polynomial $p \in \Pi_n$ that interpolates a function $f$ at the points $x_0, x_1, \ldots, x_m \in \mathbb{R}$, i.e.,

$$p(x_j) = f(x_j), \quad \text{for} \quad j = 0, 1, \ldots, m.$$

   (a) Under what conditions, we have the existence and uniqueness of $p \in \Pi_n$ ? (Justify your answer.)

   (b) Describe one approach in pseudo-code form for the construction of interpolation polynomial.

4. (15 points) Let $I_w(f) = \int_0^1 w(x)f(x)dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = af(x_1)$. Find $a$ and $x_1$ in such a way that $Q(f)$ have maximum degree of exactness $r$.