臺灣大學數學系
九十八學年度碩士班甄試試題
科目：線性代數　　　2008.10.31

You should include in your answer every piece of computation and every piece of reasoning so that the corresponding partial credit could be gained.

(1) (20 %) Let V be the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) For any integer n, define $f_n(x) = \cos(x+n)$. Find the dimension of the subspace of V generated by $\{f_n(x) : n \in \mathbb{Z}\}$.

(b) For any integer n, define $g_n(x) = e^{nx}$. Find the dimension of the subspace of V generated by $\{g_n(x) : n \in \mathbb{Z}\}$.

(2) (25 %) Let V be a finite dimensional vector space and V^* be its dual space. For any subspace W, let $W^0 = \{f \in V^* : f(w) = 0 \text{ for all } w \in W\}$.

(a) Show that $(W_1 \cap W_2 \cap \cdots \cap W_k)^0 = W_1^0 + W_2^0 + \cdots + W_k^0$.

(b) Let $f_1, f_2, \ldots, f_k, g \in V^*$. Show that $\cap_{i=1}^k \ker f_i \subseteq \ker g$ if and only if g is a linear combination of $f_1, f_2, \ldots, f_k$.

(3) (25 %) Let T be a linear transformation in a finite dimensional vector space V over K. The spectrum of T is $\mathrm{Sp}(T) := \{\lambda \in K : \lambda I - T \text{ is singular}\}$.

(a) If T is invertible, show that $\lambda \in \mathrm{Sp}(T)$ if and only if $\lambda^{-1} \in \mathrm{Sp}(T^{-1})$.

(b) Let K be algebraically closed. Show that, for any polynomial $p(x) \in K[x]$, $p(\mathrm{Sp}(T)) = \mathrm{Sp}(p(T))$.

(4) (15 %) Let V be an inner product space and T be a linear transformation in V. Show that there exists $M > 0$ such that $\|Tv\| \leq M\|v\|$ for all $v \in V$,

(5) (15 %) Show that the following matrices in $M_p(\mathbb{Z}/(p))$, p a prime, are similar:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ & \cdots & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$