臺灣大學數學系
九十一學年度第一學期碩士班甄試入學試題
線性代數乙
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1.

(a) Let $A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ -2 & 3 & 1 & -6 \\ -1 & 4 & -4 & 3 \\ 5 & -8 & 4 & 0 \end{bmatrix}$. Find a factorization $A = BC$ where B is lower triangular and C is upper triangular.

(b) Let A be an invertible matrix. Show that a factorization $A = BDC$, where B is lower triangular with all main diagonal entries 1, C is upper triangular with all main diagonal entries 1, D is diagonal, is unique.

2.

Let a $2n \times 2n$ matrix be given in the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.where each block is an $n \times n$ matrix. Suppose that A is invertible and that $AC = CA$. Show that

$$\text{Det}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \text{Det}\,(AD - CB)$$

(Hint: You can use the following fact without proof: If $C = 0$,then

$$\text{Det}\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\text{Det}\,A)(\text{Det}\,D).)$$

3.

Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V.

(a) Show that there exists k such that Im$T^k =$ImT^m and Ker$T^k =$KerT^m for all $m \geq k$.

(b) Show that there exists n such that Ker$T^n \cap$Im$T^n = (0)$.

4.

Let T be a unitary linear transformation on a finite dimensional vector space V over $\mathbf{C}$ (i.e. $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in V$). Prove that T has a unitary square root U (i.e. $U^2 = T$).