國立臺灣大學數學系114學年度碩士班甄試入學筆試

線性代數

1. (**20 points.**) Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

be a matrix over $\mathbb{R}$. Find the Smith normal form of $xI_4 - A \in M(4, \mathbb{R}[x])$ and the Jordan normal form of A.

2. (**20 points.**) Let

$$A = \begin{pmatrix} 3 & -2 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

(a) Find the maximum of $X^t A X$ among all $X \in \mathbb{R}^3$ satisfying $X^t X = 1$. Give an example of X that attains the maximum.

(b) Find the minimum of $\operatorname{tr}(Y^t A Y)$ among all 3×2 matrices Y over $\mathbb{R}$ satisfying $Y^t Y = I_2$. Give an example of Y that attains the minimum.

3. (**20 points.**) Let A be an invertible matrix over $\mathbb{C}$. Prove that there is a polynomial $p(x)$ in $\mathbb{C}[x]$ such that $A^{-1} = p(A)$.

4. (**20 points.**) Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ with (positive definite) inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, respectively. Let $T : V \to W$ be a linear transformation.

(a) Prove that for each $w \in W$, there exists a unique vector in V, denoted by $T^* w$, such that $\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V$ for all $v \in V$ and show that $T^* : W \to V$ is a linear transformation.

(b) Verify that $(\operatorname{Im} T)^\perp = \ker T^*$.

(c) Prove that $\dim \ker T - \dim \ker T^* = \dim V - \dim W$.

5. (**20 points.**) Let $T : V \to V$ be a linear transformation on a finite-dimensional vector space over an infinite field F. Prove that V is T-cyclic if and only if V has only finitely many T-invariant subspaces.