臺灣大學數學系113學年度碩士班甄試筆試試題

科目：線性代數

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1. Let $A \in M(3, \mathbb{R})$ be given by

$$A = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) (**10 points.**) Find the Jordan-Chevalley decomposition of A.

(b) (**10 points.**) Compute

$$\exp A := I_3 + \sum_{k=1}^{\infty} \frac{A^k}{k!}.$$

2. Let V be the space of all polynomials in x over $\mathbb{R}$ of degree ≤ 2. Let an inner product on V be defined by

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) (**10 points.**) Find a polynomial $k(x,t)$ in x and t such that

$$f(x) = \int_{-1}^{1} k(x,t)f(t)\,dt$$

for all $f \in V$.

(b) (**10 points.**) Let $T : V \to V$ be the linear transformation defined by $T(a_2x^2 + a_1x + a_0) = 2a_2x + a_1$. Find the linear transformation $T^* : V \to V$ such that $\langle T(f), g\rangle = \langle f, T^*(g)\rangle$ for all $f, g \in V$.

3. (**20 points.**) Let $V = M(n, \mathbb{R})$ be the vector space of all $n \times n$ matrices over $\mathbb{R}$ and $f : V \to \mathbb{R}$ be a linear transformation. Assume that $f(AB) = f(BA)$ for all $A, B \in V$ and $f(I_n) = n$, where I_n is the identity matrix in V. Prove that f is the trace function. (*Hint*: Consider the cases $A = E_{ij}$ and $B = E_{k\ell}$ for various E_{ij} and $E_{k\ell}$. Here E_{ij} denotes the matrix whose (i,j)-entry is 1 and whose other entries are 0.)

4. Let U and V be finite-dimensional vector spaces, and U^* and V^* be their dual spaces, respectively. For a linear transformation $T : U \to V$, define $T^* : V^* \to U^*$ by $(T^*f)(u) = f(Tu)$ for $f \in V^*$ and $u \in U$.

(a) (**10 points.**) Prove that T is injective if and only if T^* is surjective.

(b) (**10 points.**) Prove that T is surjective if and only if T^* is injective.

5. (**20 points.**) Let V a finite-dimensional vector space over a field F and $T : V \to V$ be a linear transformation. Assume that $f(x)$ and $g(x)$ are two relatively prime polynomials in $F[x]$. Prove that $\ker(f(T)g(T)) = \ker f(T) \oplus \ker g(T)$. (Here for a linear transformation S, we let $\ker S$ denote the kernel of S.)