臺灣大學數學系 109 學年度碩士班甄試試題

科目：線性代數

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1. Let A be a 4×4 real symmetric matrix. Suppose that 1 and 2 are eigenvalues of A and the eigenspace for the eigenvalue 2 is 3-dimensional. Assume that $(1, -1, -1, 1)^t$ is an eigenvector for the eigenvalue 1. (Here v^t denotes the transpose of v.)

(a) Find an orthonormal basis for the eigenspace for the eigenvalue 2 of A. (**10 points.**)

(b) Find Av, where $v = (1, 0, 0, 0)^t$. (**10 points.**)

2. Let A be a real $n \times n$ matrix. Prove that

$$\operatorname{rank}(A^2) - \operatorname{rank}(A^3) \leq \operatorname{rank}(A) - \operatorname{rank}(A^2).$$

(**10 points.**)

3. Let V be a vector space of finite dimension over $\mathbb{R}$ and S, T, and U be subspaces of V. Prove or disprove (by giving counterexamples) the following statements:

(a) $\dim(S + T) = \dim S + \dim T - \dim(S \cap T)$. (**10 points.**)

(b) $\dim(S + T + U) = \dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U)$. (**10 points.**)

4. (a) Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. Compute $\exp A$. (**10 points.**)

(b) Prove that $\det(\exp A) = \exp(\operatorname{tr} A)$ for $A \in M(n, \mathbb{C})$. (**10 points.**)

(c) Prove or disprove (by giving counterexamples) that if A is nilpotent, then so is $\exp A - I_n$. Here a matrix M is said to be nilpotent if $M^k = 0$ for some positive integer k and I_n is the identity matrix of size n. (**10 points.**)

5. Let U and V be finite-dimensional vector spaces, and U^* and V^* be their dual spaces, respectively. For a linear transformation $T : U \to V$, define $T^* : V^* \to U^*$ by $(T^* f)(u) = f(Tu)$ for $f \in V^*$ and $u \in U$.

(a) Prove that T is injective if and only if T^* is surjective. (**10 points.**)

(b) Prove that T is surjective if and only if T^* is injective. (**10 points.**)