臺灣大學數學系 107 學年度碩士班甄試試題

科目：線性代數

2017.10.20

(1) (20 points) Let $A = \begin{pmatrix} -1 & 3 & -2 \\ 2 & 3 & 0 \\ 11 & -6 & 7 \end{pmatrix}$. Find the lower triangular Jordan canonical form of A. Please compute $\exp(tA)$ and derive the general solution to $x'(t) = A\,x(t)$, where $x(t)$ is a 3-dimensional column vector.

(2) (20 points) Let V be an n-dimensional complex vector space, and $T : V \to V$ be an invertible linear map such that $T^2 = 1$. (a) Show that T is diagonalizable, (b) Let S be the vector space of linear transformations from V to V that commute with T. Please express $\dim_{\mathbb{C}} S$ in terms of n and the trace of T.

(3) (20 points) Let $A = (A_{ij})$ be a real invertible skew-symmetric $2n \times 2n$ matrix.
(a) Show that all eigenvalues of A are pure imaginary.
(b) Define the Pfaffian $Pf(A)$ of A by

$$Pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(2n-1),\sigma(2n)} \,.$$

Let B be any real $2n \times 2n$ matrix. Show that $Pf(BAB^T) = Pf(A)\det(B)$.
(c) Assuming the fact that there exists a real orthogonal $2n \times 2n$ matrix O such that

$$OAO^T = \operatorname{diag}\left\{ \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & m_n \\ -m_n & 0 \end{pmatrix} \right\},$$

where $m_i \in \mathbb{R}$ for $i = 1, \cdots, n$. Show that $\det(A) = Pf(A)^2$.

(4) (20 points) Let $A, B \in M_n(\mathbb{C})$ be $n \times n$ complex matrices. Show that A and B are simultaneously triangularizable (*i.e.* there exists an invertible matrix $P \in GL_n(\mathbb{C})$ such that PAP^{-1} and PBP^{-1} are both upper triangular) if A and B commute.
Hint: Let λ be one of the eigenvalues of A. Try to show $B(\ker(A - \lambda I)) \subset \ker(A - \lambda I)$.

(5) (20 points) Show that

$$\begin{vmatrix} X_0 & X_1 & X_2 & \dots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \dots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \dots & X_0 \end{vmatrix} = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \zeta^{jk} X_k \right)$$

where ζ is a primitive n-th root of unity.

Hint: You may first compute, for example, $\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_3 & X_0 & X_1 & X_2 \\ X_2 & X_3 & X_0 & X_1 \\ X_1 & X_2 & X_3 & X_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 \\ 1 & \zeta^2 & \zeta^4 & \zeta^6 \\ 1 & \zeta^3 & \zeta^6 & \zeta^9 \end{pmatrix}$.