臺灣大學數學系 106 學年度碩士班甄試試題

科目：線性代數

2016.10.21

1. (20%) Let $A \in M_{n\times n}(F)$ where F is a field.
   (a) Show that if k is the largest integer such that some $k \times k$ submatrix of A has a nonzero determinant, then $\text{rank}(A) = k$.
   (b) If A is nilpotent of index m (that is, $A^m = 0$ but $A^{m-1} \neq 0$), and if, for each vector v in F^n, W_v is defined to be the subspace generated by $v, Av, \ldots, A^{m-1}v$, how large can the dimension of W_v be? (Justify your answer.)

2. (30%)
   (a) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find the general solution to the system of differential equations

$$\frac{dX}{dt} = AX, \text{ where } X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

   where for each i, $x_i(t)$ is a differentiable real-valued function of the real variable t.
   (b) Let V be the space of all real polynomials having degree less than 4 with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$. Let T be a linear operator on V defined by $T(f(x)) = f'(x) + 3f(x)$.
   Use the Gram-Schmidt process to replace $\beta = \{1, 1+x, x+x^2, x^2+x^3\}$ by an orthonormal basis for V and find the matrix representation of the adjoint T^* of T in this orthonormal basis.

3. (30%)
   (a) Let $A \in M_{n\times n}(\mathbb{R})$. Show that there exists an orthogonal matrix Q and a positive semi-definite symmetric matrix P such that $A = QP$.
   (b) Let V be a finite-dimensional vector space over $\mathbb{C}$ and T be a linear operator on V. Show that T is normal if and only if its adjoint $T^* = g(T)$ for some polynomial $g(x) \in \mathbb{C}[x]$.

4. (20 %) Let $T \in \text{End}_{\mathbb{C}}(V)$ for a finite-dimensional $\mathbb{C}$-vector space V.
   (a) Show that we have an expression of T as $T = S + N$ with $S, N \in \text{End}_{\mathbb{C}}(V)$, such that S is diagonalisable, N is nilpotent and $SN = NS$.
   (b) Show that both S and N are uniquely defined by these conditions.
   (c) Show that there is a polynomial $p(x) \in \mathbb{C}[x]$ with $p(0) = 0$ such that $S = p(T)$.