臺灣大學數學系 105 學年度碩士班甄試試題

科目：線性代數

2015. 10. 23

There are five problems $1 \sim 5$ in total; some problems contain sub-problems, indexed by (a), (b), etc.

1. [20%] Prove the Cayley-Hamilton theorem: Let V be a finite-dimensional vector space and $T : V \to V$ a linear transformation with characteristic polynomial $p(x)$. Then $p(T) = 0$.

2. Let F be a field and $A \in M_{m\times n}(F)$ be an m by n matrix.

   (a) [10%] Show that the row rank of A equals the column rank of A.

   (b) [10%] Denote by A^t the transport of A and let r be the row rank of A. Show that AA^t is of rank r.

3. [20%] Let F be a field and $\{A_i \in M_n(F) \,|\, i \in I\}$ be a collection of n by n matrices. (I is a set; it might be finite or infinite.) Suppose that A_i are diagonalizable for all $i \in I$ and $A_iA_j = A_jA_i$ for any $i, j \in I$. Show that there exists an invertible matrix $P \in M_n(F)$ such that PA_iP^{-1} are diagonal for all $i \in I$.

4. [20%] Let $A \in M_n(\mathbb{C})$ be an n by n matrix over the field of complex numbers. Denote by A^* the conjugate transport of A. Suppose $AA^* = A^*A$. Show that there exists a matrix P such that (i) $PP^* = I$, the identity matrix, and (ii) PAP^* is a diagonal matrix. (You may do the case $A = A^*$ for half credit.)

5. [20%] Let V be a finite-dimensional vector space over the field of complex numbers and $T : V \to V$ a linear transformation with characteristic polynomial $p(x)$. Suppose that $p(x) = q_1(x)q_2(x)$ for two polynomials $q_1(x)$ and $q_2(x)$ which do not have a common root. Show that there are two subspaces W_1 and W_2 of V satisfying that (i) $W_1 \cap W_2 = \{0\}$ and $V = W_1 + W_2$, (ii) $T(W_i) \subset W_i$ for each $i = 1, 2$, and (iii) regarding T as a linear transformation on W_i, it has characteristic polynomial $q_i(x)$ for $i = 1, 2$.